Calculus for the Life Sciences
Lecture Notes – Linear Differential Equations

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Introduction

Examples of linear first order differential equations

- Malthusian growth
- Radioactive decay
- Newton’s law of cooling
- Pollution in a Lake

Extend earlier techniques to find solutions
Radioactive Decay: Radioactive elements are important in many biological applications

- $^3$H (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of *E. coli* that are unable to manufacture one particular DNA base
  - Using antibiotics, one uses the radioactive signal to determine how much DNA is replicated under a particular set of experimental conditions

- Radioactive iodine is often used to detect or treat thyroid problems
- Most experiments are run so that radioactive decay is not an issue
  - $^3$H has a half-life of 12.5 yrs
  - $^{131}$I has a half-life of 8 days
Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens

- A living organism is continually changing its carbon with the environment
  - Plants directly absorb CO$_2$ from the atmosphere
  - Animals get their carbon either directly or indirectly from plants
- Gamma radiation that bombards the Earth keeps the ratio of $^{14}$C to $^{12}$C fairly constant in the atmospheric CO$_2$
- $^{14}$C stays at a constant concentration until the organism dies
**Carbon Radiodating**

**Modeling Carbon Radiodating:** Radioactive carbon, $^{14}$C, decays with a **half-life of 5730 yr**

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon
- The loss of $^{14}$C from a sample at any time $t$ is proportional to the amount of $^{14}$C remaining
- Let $R(t)$ be the dpm per gram of $^{14}$C from an ancient object
- The differential equation for a gram of $^{14}$C
  
  $$\frac{dR(t)}{dt} = -kR(t) \quad \text{with} \quad R(0) = 15.3$$

  - This differential equation has the solution
    
    $$R(t) = 15.3 e^{-kt}, \quad \text{where} \quad k = \frac{\ln(2)}{5730} = 0.000121$$
**Example Carbon Radiodating:** Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon

Find the age of this object

**Solution:** From above

\[ 5.2 = 15.3 e^{-kt} \]

\[ e^{kt} = \frac{15.3}{5.2} = 2.94 \]

\[ kt = \ln(2.94) \]

Thus, \( t = \frac{\ln(2.94)}{k} = 8915 \text{ yr} \), so the object is about 9000 yrs old
Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is *thyroxine*, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
- Result is low blood sugar causing lethargy or mood disorders and sexual dysfunction
- One treatment for hyperthyroidism is ablating the thyroid with a large dose of radioactive iodine, $^{131}$I
  - The thyroid concentrates iodine brought into the body
  - The $^{131}$I undergoes both β and γ radioactive decay, which destroys tissue
  - Patient is given medicine to supplement the loss of thyroxine
Hyperthyroidism: Treatment

Based upon the thyroid condition and body mass, a standard dose ranges from 110-150 mCi (milliCuries), given in a special “cocktail”

- It is assumed that almost 100% of the $^{131}$I is absorbed by the blood from the gut
- The thyroid uptakes 30% of this isotope of iodine, peaking around 3 days
- The remainder is excreted in the urine
- The half-life of $^{131}$I is 8 days, so this isotope rapidly decays

Still the patient must remain in a designated room for 3-4 days for this procedure, so that he or she does not irradiate the public from his or her treatment
Hyperthyroidism Example: Assume that a patient is given a 120 mCi cocktail of $^{131}$I and that 30% is absorbed by the thyroid.

- Find the amount of $^{131}$I in the thyroid (in mCi), if the patient is released four days after swallowing the radioactive cocktail.
- Calculate how many mCi is the patient’s thyroid retains after 30 days, assuming that it was taken up by the thyroid and not excreted in the urine.
Solution:

- Assume for simplicity of the model that the $^{131}\text{I}$ is immediately absorbed into the thyroid, then stays there until it undergoes radioactive decay.
- Since the thyroid uptakes 30% of the 120 mCi, assume that the thyroid has 36 mCi immediately after the procedure.
- This is an oversimplification as it takes time for the $^{131}\text{I}$ to accumulate in the thyroid.
- This allows the simple model

$$\frac{dR}{dt} = -kR(t) \quad \text{with} \quad R(0) = 36 \text{ mCi}$$
Solution (cont): The radioactive decay model is

\[
\frac{dR}{dt} = -k R(t) \quad \text{with} \quad R(0) = 36 \text{ mCi}
\]

- The solution is
  \[
  R(t) = 36 e^{-kt}
  \]

- Since the half-life of $^{131}I$ is 8 days, after 8 days there will are 18 mCi of $^{131}I$

- Thus, $R(8) = 18 = 36 e^{-8k}$, so
  \[
  e^{8k} = 2 \quad \text{or} \quad 8k = \ln(2)
  \]

- Thus, $k = \frac{\ln(2)}{8} = 0.0866 \text{ day}^{-1}$
Solution (cont): Since

\[ R(t) = 36 e^{-kt} \quad \text{with} \quad k = 0.0866 \, \text{day}^{-1} \]

- At the time of the patient’s release \( t = 4 \) days, so in the thyroid

\[ R(4) = 36 e^{-4k} = \frac{36}{\sqrt{2}} = 25.46 \, \text{mCi} \]

- After 30 days, we find in the thyroid

\[ R(30) = 36 e^{-30k} = 2.68 \, \text{mCi} \]
Hyperthyroidism

Graph of $R(t)$

$R(t)$ Remaining in Thyroid

$R(t)$ (mCi$^{131}I$)

$t$ (days)

half-life
General Solution to Linear Growth and Decay Models:
Consider
\[
\frac{dy}{dt} = ay \quad \text{with} \quad y(t_0) = y_0
\]

The solution is
\[
y(t) = y_0 e^{a(t-t_0)}
\]
Example: Linear Decay Model

Consider

$$\frac{dy}{dt} = -0.3y \quad \text{with} \quad y(4) = 12$$

The solution is

$$y(t) = 12 e^{-0.3(t-4)}$$
Solution of General Linear Model

Consider the Linear Model

$$\frac{dy}{dt} = ay + b \quad \text{with} \quad y(t_0) = y_0$$

Rewrite equation as

$$\frac{dy}{dt} = a\left(y + \frac{b}{a}\right)$$

Make the substitution $z(t) = y(t) + \frac{b}{a}$, so $\frac{dz}{dt} = \frac{dy}{dt}$ and

$z(t_0) = y_0 + \frac{b}{a}$

$$\frac{dz}{dt} = az \quad \text{with} \quad z(t_0) = y_0 + \frac{b}{a}$$
Solution of General Linear Model

The shifted model is

\[ \frac{dz}{dt} = az \quad \text{with} \quad z(t_0) = y_0 + \frac{b}{a} \]

The solution to this problem is

\[ z(t) = \left( y_0 + \frac{b}{a} \right) e^{a(t-t_0)} = y(t) + \frac{b}{a} \]

The solution is

\[ y(t) = \left( y_0 + \frac{b}{a} \right) e^{a(t-t_0)} - \frac{b}{a} \]
Example of Linear Model

Consider the Linear Model

\[ \frac{dy}{dt} = 5 - 0.2y \quad \text{with} \quad y(3) = 7 \]

Rewrite equation as

\[ \frac{dy}{dt} = -0.2(y - 25) \]

Make the substitution \( z(t) = y(t) - 25 \), so \( \frac{dz}{dt} = \frac{dy}{dt} \) and \( z(3) = -18 \)

\[ \frac{dz}{dt} = -0.2z \quad \text{with} \quad z(3) = -18 \]
Example of Linear Model  The substituted model is

\[
\frac{dz}{dt} = -0.2z \quad \text{with} \quad z(3) = -18
\]

Thus,

\[
z(t) = -18e^{-0.2(t-3)} = y(t) - 25
\]

The solution is

\[
y(t) = 25 - 18e^{-0.2(t-3)}
\]
The linear differential equation was transformed into the IVP:

\[
\frac{dy}{dt} = -0.2(y - 25), \quad \text{with} \quad y(3) = 7
\]

The graph is given by
Newton’s Law of Cooling:

- After a murder (or death by other causes), the forensic scientist takes the temperature of the body.
- Later the temperature of the body is taken again to find the rate at which the body is cooling.
- Two (or more) data points are used to extrapolate back to when the murder occurred.
- This property is known as Newtown’s Law of Cooling.
Newton’s Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature.

If \( T(t) \) is the temperature of the body, then it satisfies the differential equation

\[
\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0
\]

- The parameter \( k \) is dependent on the specific properties of the particular object (body in this case).
- \( T_e \) is the environmental temperature.
- \( T_0 \) is the initial temperature of the object.
Murder Example

Suppose that a murder victim is found at 8:30 am
The temperature of the body at that time is 30°C
Assume that the room in which the murder victim lay was a constant 22°C
Suppose that an hour later the temperature of the body is 28°C
Normal temperature of a human body when it is alive is 37°C
Use this information to determine the approximate time that the murder occurred
Murder Example

Solution: From the model for Newton’s Law of Cooling and the information that is given, if we set \( t = 0 \) to be 8:30 am, then we solve the initial value problem

\[
\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30
\]

- Make a change of variables \( z(t) = T(t) - 22 \)
- Then \( z'(t) = T'(t) \), so the differential equation above becomes

\[
\frac{dz}{dt} = -kz(t), \quad \text{with} \quad z(0) = T(0) - 22 = 8
\]

- This is the radioactive decay problem that we solved
- The solution is

\[
z(t) = 8e^{-kt}
\]
Murder Example

Solution (cont): From the solution \( z(t) = 8 e^{-kt} \), we have

\[
\begin{align*}
  z(t) &= T(t) - 22, \quad \text{so} \quad T(t) = z(t) + 22 \\
  T(t) &= 22 + 8 e^{-kt}
\end{align*}
\]

- One hour later the body temperature is 28°C

\[
T(1) = 28 = 22 + 8 e^{-k}
\]

- Solving

\[
6 = 8 e^{-k} \quad \text{or} \quad e^k = \frac{4}{3}
\]

- Thus, \( k = \ln \left(\frac{4}{3}\right) = 0.2877 \)
Murder Example

Solution (cont): It only remains to find out when the murder occurred

- At the time of death, $t_d$, the body temperature is 37°C
  \[ T(t_d) = 37 = 22 + 8e^{-kt_d} \]

- Thus,
  \[ 8e^{-kt_d} = 37 - 22 = 15 \quad \text{or} \quad e^{-kt_d} = \frac{15}{8} = 1.875 \]

- This gives $-kt_d = \ln(1.875)$ or
  \[ t_d = -\frac{\ln(1.875)}{k} = -2.19 \]

- The murder occurred about 2 hours 11 minutes before the body was found, which places the time of death around 6:19 am
Cooling Tea: We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk

- Begin with \( \frac{4}{5} \) cup of boiling hot tea, \( T(0) = 100^\circ C \)
- Assume the tea cools according to Newton’s law of cooling

\[
\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T_e = 20^\circ C
\]

- \( k \) is the cooling constant based on the properties of the cup to be calculated
- a. In the first scenario, you let the tea cool for 5 minutes, then add \( \frac{1}{5} \) cup of cold milk, \( 5^\circ C \)
Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
- Determine the value of $k$, which we assume stays the same in this problem
- Mix in the milk, assuming that the temperature mixes perfectly in proportion to the volume of the two liquids
- In the second case, add $\frac{1}{5}$ cup of cold milk, 5°C, immediately and mix it thoroughly
- Find how long until each cup of tea reaches a temperature of 70°C
Solution of Cooling Tea: Find the rate constant $k$ for
\[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 100 \quad \text{and} \quad T(2) = 95 \]

- Let $z(t) = T(t) - 20$, so $z(0) - T(0) - 20 = 80$
- Since $z'(t) = T'(t)$, the initial value problem becomes
\[ \frac{dz}{dt} = -k z(t), \quad z(0) = 80 \]

- The solution is
\[ z(t) = 80 e^{-kt} = T(t) - 20 \]

- Thus,
\[ T(t) = 80 e^{-kt} + 20 \]
Solution (cont): The solution is

\[ T(t) = 80e^{-kt} + 20 \]

- Since \( T(2) = 95 \),
  \[ 95 = 80e^{-2k} + 20 \quad \text{or} \quad e^{2k} = \frac{80}{75} \]
  \[ k = \frac{\ln\left(\frac{80}{75}\right)}{2} = 0.03227 \]
- Find the temperature at 5 min
  \[ T(5) = 80e^{-5k} + 20 = 88.1^\circ\text{C} \]
- Now mix the \( \frac{4}{5} \) cup of tea at 88.1°C with the \( \frac{1}{5} \) cup of milk at 5°C, so
  \[ T_+(5) = 88.1 \left( \frac{4}{5} \right) + (5\frac{1}{5}) = 71.5^\circ\text{C} \]
Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

\[ T_+(5) = 71.5^\circ C \]

- The new initial value problem is

\[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(5) = 71.5^\circ C \]

- With the same substitution, \( z(t) = T(t) - 20 \),

\[ \frac{dz}{dt} = -kz, \quad z(5) = 51.5 \]

- This has the solution

\[ z(t) = 51.5e^{-k(t-5)} = T(t) - 20 \]
Solution (cont): For the first scenario, the temperature satisfies

\[ T(t) = 51.5e^{-k(t-5)} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 51.5e^{-k(t-5)} + 20 \]

- Thus,

\[ e^{k(t-5)} = \frac{51.5}{50} \]

- It follows that \( k(t - 5) = \ln(51.5/50) \), so

\[ t = 5 + \frac{\ln(51.5/50)}{k} = 5.92 \text{ min} \]
Solution (cont): For the second scenario, we mix the tea and milk, so

\[ T(0) = 100 \left( \frac{4}{5} \right) + 5 \left( \frac{1}{5} \right) = 81^\circ C \]

- The new initial value problem is
  \[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 81^\circ C \]

- With \( z(t) = T(t) - 20 \),
  \[ \frac{dz}{dt} = -k z(t), \quad z(0) = 61 \]

- This has the solution
  \[ z(t) = 61e^{-kt} = T(t) - 20 \]
Solution (cont): For the second scenario, the solution is

\[ T(t) = 61 e^{-kt} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 61e^{-kt} + 20 \]

- Thus,

\[ e^{kt} = \frac{61}{50} \]

- Since \( kt = \ln \left( \frac{61}{50} \right) \),

\[ t = \frac{\ln(61/50)}{k} = 6.16 \text{ min} \]

- Waiting to pour in the milk for 5 minutes, saves about 15 seconds in cooling time
Graph of Cooling Tea

Newton’s Law of Cooling
Pollution in a Lake: Introduction

- One of the most urgent problems in modern society is how to reduce the pollution and toxicity of our water sources.
- These are very complex issues that require a multidisciplinary approach and are often politically very intractable because of the key role that water plays in human society and the many competing interests.
- Here we examine a very simplistic model for pollution of a lake.
- The model illustrates some basic elements from which more complicated models can be built and analyzed.
Pollution in a Lake: Problem set up

- Consider the scenario of a new pesticide that is applied to fields upstream from a clean lake with volume $V$
- Assume that a river receives a constant amount of this new pesticide into its water, and that it flows into the lake at a constant rate, $f$
- This assumption implies that the river has a constant concentration of the new pesticide, $p$
- Assume that the lake is well-mixed and maintains a constant volume by having a river exiting the lake with the same flow rate, $f$, of the inflowing river
Diagram for Lake Problem Design a model using a linear first order differential equation for the concentration of the pesticide in the lake, \( c(t) \)

\[
f: \text{flow rate} \\
p: \text{pollutant} \\
V: \text{Volume} \\
c(t): \text{concentration of pollutant in the lake}
\]
Differential Equation for Pollution in a Lake

- Set up a differential equation that describes the mass balance of the pollutant
- **The change in amount of pollutant = Amount entering - Amount leaving**
  - The amount entering is simply the concentration of the pollutant, $p$, in the river times the flow rate of the river, $f$
  - The amount leaving has the same flow rate, $f$
  - Since the lake is assumed to be well-mixed, the concentration in the outflowing river will be equal to the concentration of the pollutant in the lake, $c(t)$
  - The product $f c(t)$ gives the amount of pollutant leaving the lake per unit time
Pollution in a Lake

Differential Equations for Amount and Concentration of Pollutant

- The change in amount of pollutant satisfies the model

\[
\frac{da(t)}{dt} = fp - fc(t)
\]

- Since the lake maintains a constant volume \( V \), then

\( c(t) = a(t)/V \), which also implies that \( c'(t) = a'(t)/V \)

- Dividing the above differential equation by the volume \( V \),

\[
\frac{dc(t)}{dt} = \frac{f}{V}(p - c(t))
\]

- If the lake is initially clean, then \( c(0) = 0 \)
Solution of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

\[ \frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0 \]

- This DE should remind you of Newton’s Law of Cooling with \( f/V \) acting like \( k \) and \( p \) acting like \( T_e \)
- Make the substitution, \( z(t) = c(t) - p \), so \( z'(t) = c'(t) \)
- The initial condition becomes \( z(0) = c(0) - p = -p \)
- The initial value problem in \( z(t) \) becomes,

\[ \frac{dz(t)}{dt} = -\frac{f}{V}z(t), \quad \text{with} \quad z(0) = -p \]
Solution of the Differential Equation (cont): Since

\[
\frac{dz(t)}{dt} = - \frac{f}{V} z(t), \quad \text{with} \quad z(0) = -p
\]

- The solution to this problem is

\[
z(t) = -p e^{-\frac{ft}{V}} = c(t) - p
\]

- The exponential decay in this solution shows

\[
\lim_{{t \to \infty}} c(t) = p
\]

- This is exactly what you would expect, as the entering river has a concentration of \(p\)
Example: Pollution in a Lake Part 1

1. Suppose that you begin with a 10,000 m$^3$ clean lake.
2. Assume the river entering has a flow of 100 m$^3$/day and the concentration of some pesticide in the river is measured to have a concentration of 5 ppm (parts per million).
3. Form the differential equation describing the concentration of pollutant in the lake at any time $t$ and solve it.
4. Find out how long it takes for this lake to have a concentration of 2 ppm.
Solution: This example follows the model derived above, so the differential equation for the concentration of pollutant is

\[
\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0
\]

With \( V = 10,000 \), \( f = 100 \), and \( p = 5 \),

\[
\frac{dc(t)}{dt} = -\frac{100}{10000}(c(t) - 5) \quad \text{with} \quad c(0) = 0
\]

so

\[
\frac{dc(t)}{dt} = -0.01(c(t) - 5) \quad \text{with} \quad c(0) = 0
\]
Example: Pollution in a Lake

Solution: The concentration of pollutant satisfies:

\[ \frac{dc(t)}{dt} = -0.01(c(t) - 5) \quad \text{with} \quad c(0) = 0 \]

- Let \( z(t) = c(t) - 5 \), then the differential equation becomes,

\[ \frac{dz}{dt} = -0.01z(t), \quad \text{with} \quad z(0) = -5 \]

- The solution is

\[ z(t) = -5 e^{-0.01t} = c(t) - 5 \]

- So

\[ c(t) = 5 - 5 e^{-0.01t}. \]
Example: Pollution in a Lake

Solution (cont): The concentration of pollutant in the lake is

\[ c(t) = 5 \left( 1 - e^{-0.01t} \right) \]

- Find how long it takes for the concentration to reach 2 ppm, so

\[ 2 = 5 - 5 e^{-0.01t} \quad \text{or} \quad 5 e^{-0.01t} = 3 \]

- Thus,

\[ e^{0.01t} = \frac{5}{3} \]

- Solving this for \( t \), we obtain

\[ t = 100 \ln \left( \frac{5}{3} \right) = 51.1 \text{ days} \]
Example: Pollution in a Lake Part 2

- Suppose that when the concentration reaches 4 ppm, the pesticide is banned.
- For simplicity, assume that the concentration of pesticide drops immediately to zero in the river.
- Assume that the pesticide is not degraded or lost by any means other than dilution.
- Find how long until the concentration reaches 1 ppm.
Example: Pollution in a Lake

Solution: The new initial value problem becomes

\[ \frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4 \]

- This problem is in the form of a radioactive decay problem
- This has the solution

\[ c(t) = 4e^{-0.01t} \]

- To find how long it takes for the concentration to return to 1 ppm, solve the equation

\[ 1 = 4e^{-0.01t} \quad \text{or} \quad e^{0.01t} = 4 \]

- Solving this for \( t \)

\[ t = 100 \ln(4) = 138.6 \text{ days} \]
Pollution in a Lake: Complications

The above discussion for pollution in a lake fails to account for many significant complications:

- There are considerations of degradation of the pesticide, stratification in the lake, and uptake and reentering of the pesticide through interaction with the organisms living in the lake.

- The river will vary in its flow rate, and the leaching of the pesticide into river is highly dependent on rainfall, ground water movement, and rate of pesticide application.

- Obviously, there are many other complications that would increase the difficulty of analyzing this model.

- The next section shows numerical methods to handle more complicated models.
Example: Lake Pollution with Evaporation

- Suppose that a new industry starts up river from a lake at \( t = 0 \) days, and this industry starts dumping a toxic pollutant, \( P(t) \), into the river at a rate of 7 g/day, which flows directly into the lake.

- The flow of the river is 1000 m\(^3\)/day, which goes into the lake that maintains a constant volume of 400,000 m\(^3\).

- The lake is situated in a hot area and loses 50 m\(^3\)/day of water to evaporation (pure water with no pollutant), while the remainder of the water exits at a rate of 950 m\(^3\)/day through a river.

- Assume that all quantities are well-mixed and that there are no time delays for the pollutant reaching the lake from the river.
Example: Lake Pollution with Evaporation (cont) Part a

- Write a differential equation that describes the concentration, \( c(t) \), of the pollutant in the lake, using units of mg/m\(^3\)

- Solve the differential equation

- If a concentration of only 2 mg/m\(^3\) is toxic to the fish population, then find how long until this level is reached

- If unchecked by regulations, then find what the eventual concentration of the pollutant is in the lake, assuming constant output by the new industry
Example: Lake Pollution with Evaporation

Solution: Let $P(t)$ be the amount of pollutant

The change in amount of pollutant =

Amount entering - Amount leaving

- The change in amount is $\frac{dP}{dt}$
- The concentration is given by $c(t) = \frac{P(t)}{V}$ and $c'(t) = \frac{P'(t)}{V}$
- The amount entering is the constant rate of pollutant dumped into the river, which is given by $k = 7000$ mg/day
- The amount leaving is given by the concentration of the pollutant in the lake, $c(t)$ (in mg/m$^3$), times the flow of water out of the lake, $f = 950$ m$^3$/day
Example: Lake Pollution with Evaporation

Solution (cont): The conservation of amount of pollutant is given by the equation:

\[ \frac{dP}{dt} = k - f c(t) = 7000 - 950c(t) \]

- Evaporation concentrates the pollutant by allowing water to leave without the pollutant.
- Divide the equation above by the volume, \( V = 400,000 \text{ m}^3 \):

\[ \left( \frac{1}{V} \right) \frac{dP(t)}{dt} = \frac{k}{V} - \frac{f}{V}c(t) = \frac{7}{400} - \frac{950}{400000}c(t) \]

- The concentration equation is

\[ \frac{dc}{dt} = \frac{7}{400} - \frac{950}{400000}c(t) = -\frac{f}{V} \left( c(t) - \frac{k}{f} \right) \]
Example: Lake Pollution with Evaporation

Solution (cont): The concentration equation is

\[
\frac{dc}{dt} = -\frac{95}{40000} \left( c(t) - \frac{700}{95} \right)
\]

- Make the change of variables, \( z(t) = c(t) - \frac{700}{95} \), with \( z(0) = -\frac{700}{95} \)
- The differential equation is

\[
\frac{dz}{dt} = -\frac{95}{40000} z(t) \quad \text{with} \quad z(0) = -\frac{700}{95}
\]

- The solution is

\[
z(t) = -\frac{700}{95} e^{-95t/40000} = c(t) - \frac{700}{95}
\]
Example: Lake Pollution with Evaporation

Solution (cont): The concentration equation is

\[ c(t) = \frac{700}{95} \left( 1 - e^{-95t/40000} \right) \approx 7.368 \left( 1 - e^{-0.002375t} \right) \]

- If a concentration of 2 mg/m\(^3\) is toxic to the fish population, then find when \( c(t) = 2 \) mg/m\(^3\)
- Solve

\[ 2 = 7.368 \left( 1 - e^{-0.002375t} \right) \quad \text{or} \quad e^{0.002375t} \approx 1.3726 \]

- Thus, \( t = \frac{\ln(1.3726)}{0.002375} \approx 133.3 \) days
- The limiting concentration is

\[ \lim_{t \to \infty} c(t) = \frac{700}{95} \approx 7.368 \]
Example: Lake Pollution with Evaporation (cont) Part b

- Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time $t = 0$ days

- Write a new differential equation describing the situation following the shutdown of the industry and solve this equation

- Calculate how long it takes for the lake to return to a level that allows fish to survive
Example: Lake Pollution with Evaporation

Solution: Now $k = 0$, so the initial value problem becomes

$$\frac{dc}{dt} = -\frac{95}{40000} c(t) = -0.002375 c(t) \quad \text{with} \quad c(0) = \frac{700}{95}$$

This has the solution

$$c(t) = \frac{700}{95} e^{-0.002375 t} \approx 7.368 e^{-0.002375 t}$$

The concentration is reduced to 2 mg/m³ when

$$2 = 7.368 e^{-0.002375 t} \quad \text{or} \quad e^{0.002375 t} = 3.684$$

The lake is sufficiently clean for fish when

$$t = \frac{\ln(3.684)}{0.002375} \approx 549 \text{ days}$$