Introduction

Examples of linear first order differential equations
- Malthusian growth
- Radioactive decay
- Newton’s law of cooling
- Pollution in a Lake
- Extend earlier techniques to find solutions

Radioactive Decay

Radioactive Decay: Radioactive elements are important in many biological applications

- $^3$H (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of $E. coli$ that are unable to manufacture one particular DNA base
  - Using antibiotics, one uses the radioactive signal to determine how much DNA is replicated under a particular set of experimental conditions
- Radioactive iodine is often used to detect or treat thyroid problems
- Most experiments are run so that radioactive decay is not an issue
  - $^3$H has a half-life of 12.5 yrs
  - $^{131}$I has a half-life of 8 days
Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens

- A living organism is continually changing its carbon with the environment
  - Plants directly absorb CO₂ from the atmosphere
  - Animals get their carbon either directly or indirectly from plants
- Gamma radiation that bombards the Earth keeps the ratio of \(^{14}\text{C}\) to \(^{12}\text{C}\) fairly constant in the atmospheric CO₂
- \(^{14}\text{C}\) stays at a constant concentration until the organism dies

Modeling Carbon Radiodating: Radioactive carbon, \(^{14}\text{C}\), decays with a \textbf{half-life} of 5730 yr

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon
- The loss of \(^{14}\text{C}\) from a sample at any time \(t\) is proportional to the amount of \(^{14}\text{C}\) remaining
- Let \(R(t)\) be the dpm per gram of \(^{14}\text{C}\) from an ancient object
- The differential equation for a gram of \(^{14}\text{C}\)
  \[
  \frac{dR(t)}{dt} = -kR(t) \quad \text{with} \quad R(0) = 15.3
  \]
- This differential equation has the solution
  \[
  R(t) = 15.3e^{-kt}, \quad \text{where} \quad k = \frac{\ln(2)}{5730} = 0.000121
  \]

Example: Carbon Radiodating

Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon

Find the age of this object

Solution: From above

\[
5.2 = 15.3e^{-kt}
\]
\[
e^{kt} = \frac{15.3}{5.2} = 2.94
\]
\[
kt = \ln(2.94)
\]

Thus, \(t = \frac{\ln(2.94)}{k} = 8915\) yr, so the object is about 9000 yrs old

Hyperthyroidism

Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is thyroxine, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
- Result is low blood sugar causing lethargy or mood disorders and sexual dysfunction
- One treatment for hyperthyroidism is ablating the thyroid with a large dose of radioactive iodine, \(^{131}\text{I}\)
  - The thyroid concentrates iodine brought into the body
  - The \(^{131}\text{I}\) undergoes both \(\beta\) and \(\gamma\) radioactive decay, which destroys tissue
  - Patient is given medicine to supplement the loss of thyroxine
Hyperthyroidism: Treatment

- Based upon the thyroid condition and body mass, a standard dose ranges from 110-150 mCi (milliCuries), given in a special "cocktail"
- It is assumed that almost 100% of the $^{131}\text{I}$ is absorbed by the blood from the gut
- The thyroid uptakes 30% of this isotope of iodine, peaking around 3 days
- The remainder is excreted in the urine
- The half-life of $^{131}\text{I}$ is 8 days, so this isotope rapidly decays
- Still the patient must remain in a designated room for 3-4 days for this procedure, so that he or she does not irradiate the public from his or her treatment

Solution:

- Assume for simplicity of the model that the $^{131}\text{I}$ is immediately absorbed into the thyroid, then stays there until it undergoes radioactive decay
- Since the thyroid uptakes 30% of the 120 mCi, assume that the thyroid has 36 mCi immediately after the procedure
- This is an oversimplification as it takes time for the $^{131}\text{I}$ to accumulate in the thyroid
- This allows the simple model

$$\frac{dR}{dt} = -k R(t) \quad \text{with} \quad R(0) = 36 \text{ mCi}$$

Hyperthyroidism Example: Assume that a patient is given a 120 mCi cocktail of $^{131}\text{I}$ and that 30% is absorbed by the thyroid

- Find the amount of $^{131}\text{I}$ in the thyroid (in mCi), if the patient is released four days after swallowing the radioactive cocktail
- Calculate how many mCis the patient’s thyroid retains after 30 days, assuming that it was taken up by the thyroid and not excreted in the urine

Solution (cont): The radioactive decay model is

$$\frac{dR}{dt} = -k R(t) \quad \text{with} \quad R(0) = 36 \text{ mCi}$$

- The solution is

$$R(t) = 36 e^{-kt}$$

- Since the half-life of $^{131}\text{I}$ is 8 days, after 8 days there will be 18 mCi of $^{131}\text{I}$
- Thus, $R(8) = 18 = 36 e^{-8k}$, so

$$e^{8k} = 2 \quad \text{or} \quad 8k = \ln(2)$$

- Thus, $k = \frac{\ln(2)}{8} = 0.0866 \text{ day}^{-1}$
Solution (cont): Since

\[ R(t) = 36e^{-kt} \quad \text{with} \quad k = 0.0866 \text{ day}^{-1} \]

- At the time of the patient’s release \( t = 4 \) days, so in the thyroid

\[ R(4) = 36e^{-4k} = \frac{36}{\sqrt{2}} = 25.46 \text{ mCi} \]

- After 30 days, we find in the thyroid

\[ R(30) = 36e^{-30k} = 2.68 \text{ mCi} \]

**Graph of \( R(t) \)**

**Solution of Linear Growth and Decay Models**

**General Solution to Linear Growth and Decay Models:**

Consider

\[ \frac{dy}{dt} = ay \quad \text{with} \quad y(t_0) = y_0 \]

The solution is

\[ y(t) = y_0e^{at-t_0} \]

**Example: Linear Decay Model:** Consider

\[ \frac{dy}{dt} = -0.3y \quad \text{with} \quad y(4) = 12 \]

The solution is

\[ y(t) = 12e^{-0.3(t-4)} \]
Solution of General Linear Model

Consider the Linear Model
\[ \frac{dy}{dt} = ay + b \quad \text{with} \quad y(t_0) = y_0 \]

Rewrite equation as
\[ \frac{dy}{dt} = a \left( y + \frac{b}{a} \right) \]

Make the substitution \( z(t) = y(t) + \frac{b}{a} \), so \( \frac{dz}{dt} = \frac{dy}{dt} \) and
\[ z(t_0) = y_0 + \frac{b}{a} \]

The substituted model is
\[ \frac{dz}{dt} = az \quad \text{with} \quad z(t_0) = y_0 + \frac{b}{a} \]

The solution is
\[ y(t) = \left( y_0 + \frac{b}{a} \right) e^{a(t-t_0)} - \frac{b}{a} \]

Example of Linear Model

Consider the Linear Model
\[ \frac{dy}{dt} = 5 - 0.2y \quad \text{with} \quad y(3) = 7 \]

Rewrite equation as
\[ \frac{dy}{dt} = -0.2(y - 25) \]

Make the substitution \( z(t) = y(t) - 25 \), so \( \frac{dz}{dt} = \frac{dy}{dt} \) and
\[ z(3) = -18 \]

The substituted model is
\[ \frac{dz}{dt} = -0.2z \quad \text{with} \quad z(3) = -18 \]

Thus,
\[ z(t) = -18e^{-0.2(t-3)} = y(t) - 25 \]

The solution is
\[ y(t) = 25 - 18e^{-0.2(t-3)} \]
Example of Linear Model:

The linear differential equation was transformed into the IVP:

\[
\frac{dy}{dt} = -0.2(y - 25), \quad \text{with} \quad y(3) = 7
\]

The graph is given by:

![Graph](image)

Newton's Law of Cooling:

Newton’s Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature:

- If \( T(t) \) is the temperature of the body, then it satisfies the differential equation

\[
\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0
\]

- The parameter \( k \) is dependent on the specific properties of the particular object (body in this case)
- \( T_e \) is the environmental temperature
- \( T_0 \) is the initial temperature of the object

Murder Example:

- Suppose that a murder victim is found at 8:30 am
- The temperature of the body at that time is 30°C
- Assume that the room in which the murder victim lay was a constant 22°C
- Suppose that an hour later the temperature of the body is 28°C
- Normal temperature of a human body when it is alive is 37°C
- Use this information to determine the approximate time that the murder occurred
Murder Example

Solution: From the model for Newton’s Law of Cooling and the information that is given, if we set \( t = 0 \) to be 8:30 am, then we solve the initial value problem

\[
\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30
\]

- Make a change of variables \( z(t) = T(t) - 22 \)
- Then \( z'(t) = T'(t) \), so the differential equation above becomes

\[
\frac{dz}{dt} = -kz(t), \quad \text{with} \quad z(0) = T(0) - 22 = 8
\]

- This is the radioactive decay problem that we solved
- The solution is

\[
z(t) = 8e^{-kt}
\]

Joseph M. Mahaffy, ⟨jmahaffy@mail.sdsu.edu⟩ — (25/58)

Solution (cont): From the solution \( z(t) = 8e^{-kt} \), we have

\[
z(t) = T(t) - 22, \quad \text{so} \quad T(t) = z(t) + 22
\]

- One hour later the body temperature is 28°C

\[
T(1) = 28 = 22 + 8e^{-k}
\]

- Solving

\[
6 = 8e^{-k} \quad \text{or} \quad e^k = \frac{4}{3}
\]

- Thus, \( k = \ln\left(\frac{4}{3}\right) = 0.2877 \)

Joseph M. Mahaffy, ⟨jmahaffy@mail.sdsu.edu⟩ — (26/58)

Cooling Tea

Solution (cont): We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk

- Begin with \( \frac{4}{5} \) cup of boiling hot tea, \( T(0) = 100°C \)
- Assume the tea cools according to Newton’s law of cooling

\[
\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T_e = 20°C
\]

- \( k \) is the cooling constant based on the properties of the cup to be calculated
  - a. In the first scenario, you let the tea cool for 5 minutes, then add \( \frac{1}{5} \) cup of cold milk, 5°C

Joseph M. Mahaffy, ⟨jmahaffy@mail.sdsu.edu⟩ — (28/58)
Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
- Determine the value of \( k \), which we assume stays the same in this problem
- Mix in the milk, assuming that the temperature mixes perfectly in proportion to the volume of the two liquids
- In the second case, add \( \frac{1}{5} \) cup of cold milk, 5°C, immediately and mix it thoroughly
- Find how long until each cup of tea reaches a temperature of 70°C

Solution of Cooling Tea: Find the rate constant \( k \) for

\[
\frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 100 \quad \text{and} \quad T(2) = 95
\]

- Let \( z(t) = T(t) - 20 \), so \( z(0) - T(0) - 20 = 0 \)
- Since \( z'(t) = T'(t) \), the initial value problem becomes

\[
\frac{dz}{dt} = -kz, \quad z(0) = 80
\]

- The solution is

\[
z(t) = 80e^{-kt} = T(t) - 20
\]

- Thus,

\[
T(t) = 80e^{-kt} + 20
\]

Solution (cont): The solution is

\[
T(t) = 80e^{-kt} + 20
\]

- Since \( T(2) = 95 \),

\[
95 = 80e^{-2k} + 20 \quad \text{or} \quad e^{2k} = \frac{80}{75}
\]

- Find the temperature at 5 min

\[
T(5) = 80e^{-5k} + 20 = 88.1°C
\]

- Now mix the \( \frac{4}{5} \) cup of tea at 88.1°C with the \( \frac{1}{5} \) cup of milk at 5°C, so

\[
T_+(5) = 88.1 \left( \frac{4}{5} \right) + \left( \frac{1}{5} \right) = 71.5°C
\]

Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

\[
T_+(5) = 71.5°C
\]

- The new initial value problem is

\[
\frac{dT}{dt} = -k(T(t) - 20), \quad T(5) = 71.5°C
\]

- With the same substitution, \( z(t) = T(t) - 20 \),

\[
\frac{dz}{dt} = -kz, \quad z(5) = 51.5
\]

- This has the solution

\[
z(t) = 51.5e^{-k(t-5)} = T(t) - 20
\]
Solution (cont): For the first scenario, the temperature satisfies

\[ T(t) = 51.5e^{-k(t-5)} + 20 \]

- To find when the tea is 70°C, solve
  \[ 70 = 51.5e^{-k(t-5)} + 20 \]
- Thus,
  \[ e^{k(t-5)} = \frac{51.5}{50} \]
- It follows that \( k(t-5) = \ln(51.5/50) \), so
  \[ t = 5 + \frac{\ln(51.5/50)}{k} = 5.92 \text{ min} \]

Waiting to pour in the milk for 5 minutes, saves about 15 seconds in cooling time.

Solution (cont): For the second scenario, we mix the tea and milk, so

\[ T(0) = 100 \left( \frac{4}{5} \right) + 5 \left( \frac{1}{5} \right) = 81°C \]

- The new initial value problem is
  \[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 81°C \]
- With \( z(t) = T(t) - 20 \),
  \[ \frac{dz}{dt} = -k z(t), \quad z(0) = 61 \]
- This has the solution
  \[ z(t) = 61e^{-kt} = T(t) - 20 \]

Graph of Cooling Tea
Pollution in a Lake: Introduction

- One of the most urgent problems in modern society is how to reduce the pollution and toxicity of our water sources.
- These are very complex issues that require a multidisciplinary approach and are often politically very intractable because of the key role that water plays in human society and the many competing interests.
- Here we examine a very simplistic model for pollution of a lake.
- The model illustrates some basic elements from which more complicated models can be built and analyzed.

Diagram for Lake Problem
Design a model using a linear first order differential equation for the concentration of the pesticide in the lake, $c(t)$.

$\begin{align*}
f & : \text{flow rate} \\
p & : \text{pollutant} \\
V & : \text{Volume} \\
c(t) & : \text{concentration of pollutant in the lake}
\end{align*}$

Differential Equation for Pollution in a Lake

- Set up a differential equation that describes the mass balance of the pollutant.
- The change in amount of pollutant = Amount entering - Amount leaving
- The amount entering is simply the concentration of the pollutant, $p$, in the river times the flow rate of the river, $f$.
- The amount leaving has the same flow rate, $f$.
- Since the lake is assumed to be well-mixed, the concentration in the outflowing river will be equal to the concentration of the pollutant in the lake, $c(t)$.
- The product $f \cdot c(t)$ gives the amount of pollutant leaving the lake per unit time.
Differential Equations for Amount and Concentration of Pollutant

- The change in amount of pollutant satisfies the model
  \[ \frac{da(t)}{dt} = fp - fc(t) \]

- Since the lake maintains a constant volume \( V \), then \( c(t) = a(t)/V \), which also implies that \( c'(t) = a'(t)/V \)

- Dividing the above differential equation by the volume \( V \),
  \[ \frac{dc(t)}{dt} = \frac{f}{V}(p - c(t)) \]

- If the lake is initially clean, then \( c(0) = 0 \)

Solution of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

\[ \frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0 \]

- This DE should remind you of Newton’s Law of Cooling with \( f/V \) acting like \( k \) and \( p \) acting like \( T_e \)

- Make the substitution, \( z(t) = c(t) - p \), so \( z'(t) = c'(t) \)

- The initial condition becomes \( z(0) = c(0) - p = -p \)

- The initial value problem in \( z(t) \) becomes,
  \[ \frac{dz(t)}{dt} = -\frac{f}{V}z(t), \quad \text{with} \quad z(0) = -p \]

Solution of the Differential Equation (cont): Since

\[ \frac{dz(t)}{dt} = -\frac{f}{V}z(t), \quad \text{with} \quad z(0) = -p \]

- The solution to this problem is
  \[ z(t) = -p e^{-\frac{ft}{V}} = c(t) - p \]

- The exponential decay in this solution shows
  \[ \lim_{t \to \infty} c(t) = p \]

- This is exactly what you would expect, as the entering river has a concentration of \( p \)

Example: Pollution in a Lake Part 1

- Suppose that you begin with a 10,000 m\(^3\) clean lake
- Assume the river entering has a flow of 100 m\(^3\)/day and the concentration of some pesticide in the river is measured to have a concentration of 5 ppm (parts per million)
- Form the differential equation describing the concentration of pollutant in the lake at any time \( t \) and solve it
- Find out how long it takes for this lake to have a concentration of 2 ppm
Example: Pollution in a Lake

Solution: This example follows the model derived above, so the differential equation for the concentration of pollutant is

$$\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0$$

With $V = 10,000$, $f = 100$, and $p = 5$,

$$\frac{dc(t)}{dt} = -\frac{100}{10000}(c(t) - 5) \quad \text{with} \quad c(0) = 0$$

so

$$\frac{dc(t)}{dt} = -0.01(c(t) - 5) \quad \text{with} \quad c(0) = 0$$

Solution (cont): The concentration of pollutant in the lake is

$$c(t) = 5 \left(1 - e^{-0.01t}\right)$$

Find how long it takes for the concentration to reach $2 \text{ ppm}$, so

$$2 = 5 - 5e^{-0.01t} \quad \text{or} \quad 5e^{-0.01t} = 3$$

Thus,

$$e^{0.01t} = \frac{5}{3}$$

Solving this for $t$, we obtain

$$t = 100 \ln \left(\frac{5}{3}\right) = 51.1 \text{ days}$$
Example: Pollution in a Lake

Solution: The new initial value problem becomes
\[
\frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4
\]
- This problem is in the form of a radioactive decay problem
- This has the solution
  \[c(t) = 4e^{-0.01t}\]
- To find how long it takes for the concentration to return to 1 ppm, solve the equation
  \[1 = 4e^{-0.01t} \quad \text{or} \quad e^{0.01t} = 4\]
- Solving this for \(t\)
  \[t = 100\ln(4) = 138.6 \text{ days}\]

Pollution in a Lake: Complications

- There are considerations of degradation of the pesticide, stratification in the lake, and uptake and reentering of the pesticide through interaction with the organisms living in the lake
- The river will vary in its flow rate, and the leaching of the pesticide into river is highly dependent on rainfall, ground water movement, and rate of pesticide application
- Obviously, there are many other complications that would increase the difficulty of analyzing this model
- The next section shows numerical methods to handle more complicated models

Example: Lake Pollution with Evaporation

- Suppose that a new industry starts up river from a lake at \(t = 0\) days, and this industry starts dumping a toxic pollutant, \(P(t)\), into the river at a rate of 7 g/day, which flows directly into the lake
- The flow of the river is 1000 m\(^3\)/day, which goes into the lake that maintains a constant volume of 400,000 m\(^3\)
- The lake is situated in a hot area and loses 50 m\(^3\)/day of water to evaporation (pure water with no pollutant), while the remainder of the water exits at a rate of 950 m\(^3\)/day through a river
- Assume that all quantities are well-mixed and that there are no time delays for the pollutant reaching the lake from the river

Example: Lake Pollution with Evaporation (cont) Part a

- Write a differential equation that describes the concentration, \(c(t)\), of the pollutant in the lake, using units of mg/m\(^3\)
- Solve the differential equation
- If a concentration of only 2 mg/m\(^3\) is toxic to the fish population, then find how long until this level is reached
- If unchecked by regulations, then find what the eventual concentration of the pollutant is in the lake, assuming constant output by the new industry
Example: Lake Pollution with Evaporation

Solution: Let \( P(t) \) be the amount of pollutant.

The change in amount of pollutant = Amount entering - Amount leaving

- The change in amount is \( \frac{dP}{dt} \)
- The concentration is given by \( c(t) = P(t)/V \) and \( c'(t) = P'(t)/V \)
- The amount entering is the constant rate of pollutant dumped into the river, which is given by \( k = 7000 \text{ mg/day} \)
- The amount leaving is given by the concentration of the pollutant in the lake, \( c(t) \) (in mg/m\(^3\)), times the flow of water out of the lake, \( f = 950 \text{ m}^3/\text{day} \)

Solution (cont): The concentration equation is

\[
\frac{dc}{dt} = -\frac{95}{40000} \left( c(t) - \frac{700}{95} \right)
\]

Make the change of variables, \( z(t) = c(t) - \frac{700}{95} \), with \( z(0) = -\frac{700}{95} \)

The differential equation is

\[
\frac{dz}{dt} = -\frac{95}{40000} z(t) \quad \text{with} \quad z(0) = -\frac{700}{95}
\]

The solution is

\[
z(t) = -\frac{700}{95} e^{-\frac{95}{40000}t} = c(t) - \frac{700}{95}
\]

Example: Lake Pollution with Evaporation

Solution (cont): The conservation of amount of pollutant is given by the equation:

\[
\frac{dP}{dt} = k - f c(t) = 7000 - 950 c(t)
\]

- Evaporation concentrates the pollutant by allowing water to leave without the pollutant
- Divide the equation above by the volume, \( V = 400,000 \text{ m}^3 \)

\[
\frac{1}{V} \frac{dP}{dt} = \frac{k}{V} - \frac{f}{V} c(t) = \frac{7}{400} - \frac{950}{400000} c(t)
\]

The concentration equation is

\[
\frac{dc}{dt} = \frac{7}{400} - \frac{95}{400000} c(t) = -\frac{f}{V} \left( c(t) - \frac{7}{f} \right)
\]

Solution (cont): The concentration equation is

\[
c(t) = \frac{700}{95} \left( 1 - e^{-95t/40000} \right) \approx 7.368 \left( 1 - e^{-0.002375t} \right)
\]

- If a concentration of 2 mg/m\(^3\) is toxic to the fish population, then find when \( c(t) = 2 \text{ mg/m}^3 \)
- Solve

\[
2 = 7.368 \left( 1 - e^{-0.002375t} \right) \quad \text{or} \quad e^{0.002375t} \approx 1.3726
\]

Thus, \( t = \frac{\ln(1.3726)}{0.002375} \approx 133.3 \text{ days} \)

The limiting concentration is

\[
\lim_{t \to \infty} c(t) = \frac{700}{95} \approx 7.368
\]
Example: Lake Pollution with Evaporation (cont) Part b

- Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time \( t = 0 \) days.
- Write a new differential equation describing the situation following the shutdown of the industry and solve this equation.
- Calculate how long it takes for the lake to return to a level that allows fish to survive.

Solution: Now \( k = 0 \), so the initial value problem becomes

\[
\frac{dc}{dt} = -\frac{95}{40000} c(t) = -0.002375 c(t) \quad \text{with} \quad c(0) = \frac{700}{95}
\]

- This has the solution

\[
c(t) = \frac{700}{95} e^{-0.002375t} \approx 7.368 e^{-0.002375t}
\]

- The concentration is reduced to 2 mg/m³ when

\[
2 = 7.368 e^{-0.002375t} \quad \text{or} \quad e^{0.002375t} = 3.684
\]

- The lake is sufficiently clean for fish when

\[
t = \frac{\ln(3.684)}{0.002375} \approx 549 \text{ days}
\]