Calculus for the Life Sciences
Lecture Notes – Limits, Continuity, and the Derivative

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Outline

1. Limits
   - Definition
   - Examples of Limit

2. Continuity
   - Examples of Continuity

3. Derivative
   - Examples of a derivative
Introduction

- Limits are central to Calculus
- Present definitions of limits, continuity, and derivative
- Sketch the formal mathematics for these definitions
- Graphically show these ideas
- Recall derivative is related to the slope of the tangent line
- Complete understanding of the definitions is beyond the scope of this course
Definition of Limit

Limits – Conceptually, the limit of a function \( f(x) \) at some point \( x_0 \) simply means that if your value of \( x \) is very close to the value \( x_0 \), then the function \( f(x) \) stays very close to some particular value.

Definition: The limit of a function \( f(x) \) at some point \( x_0 \) exists and is equal to \( L \) if and only if every “small” interval about the limit \( L \), say the interval \( (L - \epsilon, L + \epsilon) \), means you can find a “small” interval about \( x_0 \), say the interval \( (x_0 - \delta, x_0 + \delta) \), which has all values of \( f(x) \) existing in the former “small” interval about the limit \( L \), except possibly at \( x_0 \) itself.
Definition of Limit

Diagram for Definition of Limit

\[ y = f(x) \]

\[ y = L + \varepsilon \]

\[ y = L - \varepsilon \]

\[ P(x, f(x)) \]
Example of Limits: Consider \( f(x) = x^2 - x - 6 \)

- Find the limit as \( x \) approaches 1
- From either the graph or from the way you have always evaluated this quadratic function that as \( x \) approaches 1, \( f(x) \) approaches \(-6\), since \( f(1) = -6 \)

Fact: Any polynomial, \( p(x) \), has as its limit at some \( x_0 \), the value of \( p(x_0) \)
Example of Limits: Consider \( r(x) = \frac{x^2 - x - 6}{x - 3} \)

- Find the limit as \( x \) approaches 1
- If \( x \) is not 3, then this rational function reduces to \( r(x) = x + 2 \)
- So as \( x \) approaches 1, this function simply goes to 3

Fact: Any rational function, \( r(x) = \frac{p(x)}{q(x)} \), where \( p(x) \) and \( q(x) \) are polynomials with \( q(x_0) \) not zero, then the limit exists with the limit being \( r(x_0) \)
Example of Limits: Consider $r(x) = \frac{x^2-x-6}{x-3}$

- Find the limit as $x$ approaches 3
- Though $r(x)$ is not defined at $x_0 = 3$, arbitrarily “close” to 3, $r(x) = x + 2$
- So as $x$ approaches 3, this function goes to 5
- Its limit exists though the function is not defined at $x_0 = 3$
Example of Limits: Consider \( f(x) = \frac{1}{x^2} \)

- Find the limit as \( x \) approaches 0, if it exists.
- This function has a limit for any value of \( x_0 \) where the denominator is not zero.
- However, at \( x_0 = 0 \), this function is undefined.
- Thus, the graph has a vertical asymptote at \( x_0 = 0 \).
- This means that no limit exists for \( f(x) \) at \( x_0 = 0 \).
Example of Limits: Consider \( r(x) = \frac{x^2-x-2}{x-3} \)

- Find the limit as \( x \) approaches 3, if it exists
- This function has a limit for any value of \( x_0 \) where the denominator is not zero
- Since the numerator is not zero, while the denominator is zero at \( x_0 = 3 \), this function is undefined at \( x_0 = 3 \)
- The graph has a vertical asymptote at \( x_0 = 3 \)
- This means that no limit exists for \( r(x) \) at \( x_0 = 3 \)
Examples of Limits

For \( r(x) = \frac{x^2-x-2}{x-3} \)

**Fact:** Whenever you have a vertical asymptote at some \( x_0 \), then the limit fails to exist at that point.
Example of Limits: The Heaviside function is often used to specify when something is “on” or “off”

The Heaviside function is defined as

\[ H(x) = \begin{cases} 
0, & x < 0 \\
1, & x \geq 0 
\end{cases} \]

- This function clearly has the limit of 0 for any \( x < 0 \), and it has the limit of 1 for any \( x > 0 \)
- Even though this function is defined to be 1 at \( x = 0 \), it does not have a limit at \( x_0 = 0 \)
  - If you take some “small” interval about the proposed limit of 1, say \( \epsilon = 0.1 \), then all values of \( x \) near 0 must have \( H(x) \) between 0.9 and 1.1
  - But take any “small” negative \( x \) and \( H(x) = 0 \), which is not in the desired given interval
- Thus, no limit exists for \( H(x) \)
Examples of Limits

For

\[ H(x) = \begin{cases} 
0, & x < 0 \\
1, & x \geq 0 
\end{cases} \]

**Perspective:** Whenever a function is defined differently on different intervals (like the Heaviside function), check the \(x\)-values where the function changes in definition to see if the function has a limit at these \(x\)-values.
Example of Limits: Consider \( f(x) = \sqrt{x} \)

- Find the limit as \( x \) approaches 0, if it exists
- This function is not defined for \( x < 0 \), so it cannot have a limit at \( x = 0 \), though it is said to have a right-handed limit
Summary of Limits:

- Most of the functions in this course examine have limits
- Continuous portions of a function have limits
- Limits fail to exist at points $x_0$
  - At a vertical asymptote
  - When the function is defined differently on different intervals
  - Special cases like the square root function
Continuity

Closely connected to the concept of a limit is that of continuity.

Intuitively, the idea of a continuous function is what you would expect:
- If you can draw the function without lifting your pencil, then the function is continuous.

Most practical examples use functions that are continuous or at most have a few points of discontinuity.

**Definition:** A function $f(x)$ is **continuous** at a point $x_0$ if the limit exists at $x_0$ and is equal to $f(x_0)$.
Example 3: For

\[ r(x) = \frac{x^2 - x - 6}{x - 3} \]

- Though the limit exists at \( x_0 = 3 \), the function is not continuous there (function not defined at \( x = 3 \))
- This function is continuous at all other points, \( x \neq 3 \)
Examples 4 and 6: For

\[ f(x) = \frac{1}{x^2} \quad \text{and} \quad H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \]

- These functions are not continuous at \( x_0 = 0 \)
- These functions are continuous at all other points, \( x \neq 0 \)
Example:
Below is a graph of a function, $f(x)$, that is defined $x \in [-2, 2]$, except at $x = 0$.

Difficulties with this function occur at integer values.
Comparing Limits and Continuity

At $x = -1$, the function has the value $f(-1) = 1$

The function is not continuous nor does a limit exist at this point
At $x = 0$, the function is not defined

There is a vertical asymptote
At $x = 1$, the function has the value $f(1) = 4$

The function is not continuous, but the limit exists with

$$\lim_{x \to 1} f(x) = 1$$

At $x = 2$, the function is continuous with $f(2) = 3$, which also means that the limit exists
At all non-integer values of $x$ the function is continuous (hence its limit exists)

We will see that the derivative only exists at these non-integer values of $x$
Derivative

- The primary reason for the discussion above is for the proper definition of the derivative.
- The derivative at a point on a curve is the slope of the tangent line at that point.
- This motivation is what underlies the definition given below.

**Definition:** The derivative of a function $f(x)$ at a point $x_0$ is denoted $f'(x_0)$ and satisfies

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.
Example: Use the definition to find the derivative of

\[ f(x) = x^2 \]

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x
\end{align*}
\]
**Example:** Use the definition to find the derivative of

\[ f(x) = \frac{1}{x + 2}, \quad x \neq -2 \]

\[ f'(x) = \lim_{h \to 0} \frac{1}{h} \left( \frac{1}{x+h+2} - \frac{1}{x+2} \right) \]

\[ = \lim_{h \to 0} \frac{(x + 2) - (x + h + 2)}{h(x + 2 + h)(x + 2)} \]

\[ = \lim_{h \to 0} \frac{-h}{h(x + 2 + h)(x + 2)} \]

\[ = \lim_{h \to 0} \frac{-1}{(x + 2 + h)(x + 2)} \]

\[ = \frac{-1}{(x + 2)^2} \]
Clearly, we do not want to use this formula every time we need to compute a derivative.

Much of the remainder of this course will be learning easier ways to take the derivative.

In Lab, a very easy way to find derivatives is using the Maple `diff` command.