Introduction to the Derivative

The derivative is very important to Calculus
How to view the Derivative
  • Rate of Growth
  • Velocity
  • Geometric view: The Tangent Line

Graph of the heights of girls and boys ages 0 to 18
The rate of growth is the slope of the line through the data.
- The earliest years show a high rate of growth.
- Over a wide range of ages, the rate of growth is almost constant.
- The later years show the growth rate slowing.

Growth rate is the difference in heights divided by the difference in time measured in years.

The growth rate \( g(t) \) is approximated by the formula:

\[
g(t_0) = \frac{h(t_1) - h(t_0)}{t_1 - t_0}
\]

where \( t_0 \) and \( t_1 \) are successive ages with heights \( h(t_0) \) and \( h(t_1) \).

### Growth Rate for Children

**Girls age from 2 to 3**

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Height (cm)</th>
<th>Annual Growth Rate (cm/yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 = 2 )</td>
<td>( h(t_0) = 87 )</td>
<td>( g(2) = \frac{h(3) - h(2)}{3 - 2} = 7 )</td>
</tr>
<tr>
<td>( t_1 = 3 )</td>
<td>( h(t_1) = 94 )</td>
<td></td>
</tr>
</tbody>
</table>

**Boys age 3 months to 6 months**

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Height (cm)</th>
<th>Annual Growth Rate (cm/yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_0 = 0.25 ) (3 months)</td>
<td>( h(t_0) = 61 )</td>
<td></td>
</tr>
<tr>
<td>( t_1 = 0.5 ) (6 months)</td>
<td>( h(t_1) = 68 )</td>
<td>( g(0.25) = \frac{h(0.5) - h(0.25)}{0.5 - 0.25} = 28 )</td>
</tr>
</tbody>
</table>

The growth rate is higher for early years.
- Stays almost constant for many years.
- Drops almost to zero in the late teens.
Introduction to the Derivative
Derivative as a Growth Rate
Derivative as a Velocity
Juvenile Height
Puppy Growth

Example – Growth of a Puppy

Developing his Project Calculus course, David Smith measured the growth of his Golden Retriever puppy, Sassafras.

<table>
<thead>
<tr>
<th>Age (days)</th>
<th>Weight (lbs)</th>
<th>Age (days)</th>
<th>Weight (lbs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.25</td>
<td>101</td>
<td>30</td>
</tr>
<tr>
<td>10</td>
<td>4.25</td>
<td>115</td>
<td>37</td>
</tr>
<tr>
<td>20</td>
<td>5.5</td>
<td>150</td>
<td>54</td>
</tr>
<tr>
<td>30</td>
<td>7</td>
<td>195</td>
<td>65</td>
</tr>
<tr>
<td>40</td>
<td>9</td>
<td>230</td>
<td>70</td>
</tr>
<tr>
<td>50</td>
<td>11.5</td>
<td>332</td>
<td>75</td>
</tr>
<tr>
<td>60</td>
<td>15</td>
<td>436</td>
<td>77</td>
</tr>
<tr>
<td>70</td>
<td>19</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Find the average weekly growth rate of the puppy over the first 10 weeks.

**Solution:** 10 weeks is equivalent to 70 days.
The weight at day 0 is 3.25 lbs, while its 19 lbs at 10 weeks.
The average growth rate is
\[ g_{ave} = \frac{19 - 3.25}{10} = 1.575 \text{ lb/week} \]

Estimate the weekly growth rate of the puppy at age 10 weeks using the data at 70 and 101 days.

**Solution:** The weekly growth rate at 10 weeks satisfies
\[ g_{ave} = \frac{30 - 19}{(101 - 70)/7} = 2.48 \text{ lb/week} \]

What is the weekly growth rate between days 230 and 436?

**Solution:** The weekly growth rate between 230 and 436 days satisfies
\[ \frac{77 - 70}{(436 - 230)/7} = 0.238 \text{ lb/week} \]

The growth rate increases for several weeks, then slows down as the puppy matures.

Graphically we explore:
- The weight of the puppy as a function of age (days)
- The growth in weight per week

The growth curve for the weight of a puppy gives a typical **S-shaped curve**

The growth rate curve shows the weight gain accelerating for about 100 days, then slowing down.

The growth rate is computed between successive data points and associated with the midpoint of the ages
\[ (a_{ave} = (a_{i+1} + a_i)/2) \]
\[ g(a_{ave}) = 7 \times \frac{w_{i+1} - w_i}{a_{i+1} - a_i} \]

where \( a_i \) and \( w_i \) are obtained from the growth table. (Graphs follow)
Example – Growth of a Puppy

Derivative as a Velocity

- Differential Calculus – developed in the 17th century by Sir Isaac Newton and Gottfried Leibnitz to explain the physics of motion
- Velocity of an object is the change in distance divided by the change in time
  
  - If we travel 200 ft in 10 sec, then we had an average velocity of 20 ft/sec

Eadweard Muybridge – Trotting Horse

- In the 1800s, there was a controversy whether or not a trotting horse ever had all feet off of the ground
- Photographer Eadweard Muybridge developed some special photographic techniques for viewing animals and humans in motion by collecting timed sequences of still pictures
- Viewed in succession with the same intervening times, these pictures produce an animation of motion, which was a precursor to modern motion pictures
Trotting Horses

Were all feet off the ground at any time?

How fast is the horse trotting?

- Often want to determine how fast a particular animal is running or a bird flying
- How do you determine the speed of a cheetah or the velocity of a peregrine falcon?
- Above sequence of pictures has units of distance (ft) marked in the background and units of time on each frame
- Start by choosing a reference point, say the man’s head

Trotting Horse Speed

The position at \( t_0 = 0 \) sec, satisfies \( s(t_0) = 3.5 \) ft, while at \( t_1 = 0.04 \) sec, the head is at \( s(t_1) = 4.5 \) ft

\[
v(t_0) = \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{4.5 - 3.5}{0.04 - 0} = 25 \text{ ft/sec} = 17.0 \text{ mph}
\]

At \( t_2 = 0.08 \) sec, the head is at \( s(t_2) = 5.6 \) ft, so the velocity satisfies

\[
v(t_1) = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{5.6 - 4.5}{0.08 - 0.04} = 27.5 \text{ ft/sec} = 18.75 \text{ mph}
\]

This is approximately the same

Trotting Horse Average Speed

The average velocity for the entire sequence of pictures gives the best average velocity for this trotting horse

It is computed by taking the initial and final positions of the head and dividing by the total time between the frames

\[
v(t_{ave}) = \frac{s(t_f) - s(t_0)}{t_f - t_0} = \frac{11.5 - 3.5}{0.32 - 0} = 25.0 \text{ ft/sec} = 17.05 \text{ mph}
\]

Thus, the velocity is relatively constant over the short time interval of the pictures

More details

- What if the question asked the velocity of the right front foot?
- Does the right front hoof stop or move backward at any time?
- Current sequence of pictures is inadequate
- How would you answer this question?
- Probably want many more pictures at smaller time intervals
- **This is the limiting process we will undertake to find the derivative**
**Example – Ball Falling under the Influence of Gravity**

A steel ball is dropped from a height of 4 meters and has its height measured every 0.1 seconds.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Distance (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>5</td>
</tr>
<tr>
<td>0.2</td>
<td>19</td>
</tr>
<tr>
<td>0.3</td>
<td>44</td>
</tr>
<tr>
<td>0.4</td>
<td>78</td>
</tr>
</tbody>
</table>

Find the average speed of the ball over the 0.9 seconds of the experiment.

**Solution:**

\[
v_{ave} = \frac{396 - 0}{0.9 - 0} = 440.0 \text{ cm/sec}
\]

Determine the average speed of the ball between 0.5 and 0.7 seconds.

**Solution:**

\[
v_{ave} = \frac{240 - 123}{0.7 - 0.5} = 585 \text{ cm/sec}
\]

**Example – Sky Diving**

A sky diver encounters a significant amount of air resistance when free falling (and more significantly when the parachute opens), so his speed will not match the parabolic curve characteristic of the Falling Ball.

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Height (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10,000</td>
</tr>
<tr>
<td>5</td>
<td>9,633</td>
</tr>
<tr>
<td>10</td>
<td>8,797</td>
</tr>
<tr>
<td>15</td>
<td>7,811</td>
</tr>
<tr>
<td>20</td>
<td>6,791</td>
</tr>
</tbody>
</table>

- Graph the Height vs. Time
- Compute the approximate velocity using the successive rows of the table
- Graph the velocity curve
- What is the approximate velocity in miles per hour at 30 seconds into the fall?
- Can you estimate when the sky diver would hit the ground if the parachute failed to open?
**Example – Sky Diving**

**Solution:** Graph the Height vs. Time

![Graph](image)

**Solution:** Compute the first two velocity points of the graph:

\[
v(t_0) = \frac{h(t_1) - h(t_0)}{t_1 - t_0} = \frac{9,633 - 10,000}{5 - 0} = -73.4 \text{ ft/sec},
\]

\[
v(t_1) = \frac{h(t_2) - h(t_1)}{t_2 - t_1} = \frac{8,797 - 9,633}{10 - 5} = -167.2 \text{ ft/sec}.
\]

The velocity at 30 seconds is

\[
v(t_6) = \frac{3703 - 4733}{35 - 30} = -206 \text{ ft/sec},
\]

**Solution:** The graph shows the velocity of the sky diver levels off shortly after 10 sec

The **terminal velocity** is approximately the velocity at 30 sec or \(v_{term} = -206 \text{ ft/sec} \) (which is about \(-140.5 \text{ mph}\))

At 45 sec, the sky diver is at 1643 ft and traveling at \(v_{term}\)

We get, \(\text{time} = \frac{\text{distance}}{v} = \frac{1643}{206} = 8.0 \text{ sec} \) to cover the remaining 1643 ft

Hence, the sky diver would fall for about \(45 + 8 = 53 \text{ sec} \) if the parachute failed