Calculus for the Life Sciences

Lecture Notes – Introduction to Differential Equations

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Outline

Introduction

Introduction

- Differential equations frequently arise in modeling situations
- They describe population growth, chemical reactions, heat exchange, motion, and many other applications
- Differential equations are continuous analogs of discrete dynamical systems

What is a Differential Equation?

What is a Differential Equation?

- A differential equation is any equation of some unknown function that involves some derivative of the unknown function
- The classical example is Newton's Law of motion
  - The mass of an object times its acceleration is equal to the sum of the forces acting on that object
  - Acceleration is the first derivative of velocity or the second derivative of position
  - This is an example of a differential equation
- In biology, a differential equation describes a growth rate, a reaction rate, or the change in some physiological state

Malthusian Growth

Discrete Malthusian Growth Population, P_n, at time n with growth rate, r

$$P_{n+1} = P_n + rP_n$$

Rearrange the discrete Malthusian growth model

$$P_{n+1} - P_n = rP_n$$

The change in population between $(n+1)^{st}$ time and the n^{th} time is proportional to the population at the n^{th} time

Malthusian Growth

Malthusian Growth (cont) Let $P(t)$ be the population at time t

- Assume that r is the rate of change of the population per unit time per animal in the population
- Let Δt be a small interval of time, then the change in population between t and $t + \Delta t$, satisfies

$$P(t + \Delta t) - P(t) = \Delta t \cdot rP(t)$$

- Biologically, this equation says that the change (difference) in the population over a small period of time is found by taking the rate of growth times the population times the interval of time Δt
- The equation above can be rearranged to give

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = rP(t)$$

Continuous Malthusian Growth

Continuous Malthusian Growth The discrete model was given by

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = rP(t)$$

- The right hand side of the equation should remind you of the definition of the derivative
- Take the limit of $\Delta t \to 0$, so

$$\lim_{\Delta t \to 0} \frac{P(t + \Delta t) - P(t)}{\Delta t} = \frac{dP(t)}{dt} = rP(t)$$

- This is the **continuous Malthusian growth model**

Continuous Malthusian Growth

Solution of Malthusian Growth Model The Malthusian growth model

$$\frac{dP(t)}{dt} = rP(t)$$

- The rate of change of a population is proportional to the population
- Let c be an arbitrary constant, so try a solution of the form

$$P(t) = ce^{rt}$$

- Differentiating

$$\frac{dP(t)}{dt} = cre^{rt},$$

which is $rP(t)$, so satisfies the differential equation

Continuous Malthusian Growth

Solution of Malthusian Growth Model (cont) The Malthusian growth model satisfies

$$P(t) = ce^{rt}$$

- With the initial condition, $P(0) = P_0$, then the unique solution is
$$P(t) = P_0 e^{rt}$$
- Malthusian growth is often called exponential growth

Example: Malthusian Growth 1

Example: Malthusian Growth Consider the Malthusian growth model

$$\frac{dP(t)}{dt} = 0.02\,P(t) \quad \text{with} \quad P(0) = 100$$

Skip Example

- Find the solution
- Determine how long it takes for this population to double

Example: Malthusian Growth 2

Solution: The solution is given by

$$P(t) = 100\,e^{0.02t}$$

We can confirm this by computing

$$\frac{dP}{dt} = 0.02(100\,e^{0.02t}) = 0.02\,P(t),$$

so this solution satisfies the differential equation and the initial condition

The population doubles when

$$200 = 100\,e^{0.02t}$$
$$0.02t = \ln(2) \qquad \text{or} \qquad t = 50\ln(2) \approx 34.66$$

Example 2: Malthusian Growth 1

Example 2: Suppose that a culture of *Escherichia coli* is growing according to the Malthusian growth model

$$\frac{dP(t)}{dt} = rP(t) \quad \text{with} \quad P(0) = 100{,}000$$

Skip Example

- Assume the population doubles in 25 minutes
- Find the growth rate constant and the solution to this differential equation
- Compute the population after one hour

Example 2: Malthusian Growth 2

Solution: The general solution satisfies

$$P(t) = 100,000\, e^{rt}$$

- If the population doubles in 25 minutes, then
$$P(25) = 200,000 = 100,000\, e^{25r}$$
- Dividing by 100,000 and taking the logarithm of both sides
$$\ln(2) = 25\, r$$
- The growth rate constant is $r = 0.0277$
- The specific solution is given by
$$P(t) = 100,000\, e^{0.0277t}$$
- The population after one hour is
$$P(60) = 100,000e^{0.0277(60)} = 527,803$$

Applications of Differential Equations 1

Radioactive Decay: Let $R(t)$ be the amount of a radioactive substance

- Radioactive materials are often used in biological experiments and for medical applications
- Radioactive elements transition through decay into another state at a rate proportional to the amount of radioactive material present
- The differential equation is
$$\frac{dR(t)}{dt} = -k\, R(t) \quad \text{with} \quad R(0) = R_0$$
- Like the Malthusian growth model, this has an exponential solution
$$R(t) = R_0 e^{-kt}$$

Applications of Differential Equations 2

Harmonic Oscillator: A Hooke's law spring exerts a force that is proportional to the displacement of the spring

- Newton's law of motion: Mass times the acceleration equals the force acting on the mass
- Applied to biological phenomena
  - Vibrating cilia in ears
  - Stretching of actin filaments in muscle fibers
- The simplest spring-mass problem is
$$my'' = -cy \qquad \text{or} \qquad y'' + k^2 y = 0$$
- The general solution is
$$y(t) = c_1 \cos(kt) + c_2 \sin(kt),$$
where c_1 and c_2 are arbitrary constants

Applications of Differential Equations 3

Swinging Pendulum: A pendulum is a mass attached at one point so that it swings freely under the influence of gravity

Newton's law of motion (ignoring resistance) gives the differential equation

$$my'' + g\sin(y) = 0,$$

where y is the angle of the pendulum, m is the mass of the bob of the pendulum, and g is the gravitational constant

This problem does not have an easily expressible solution

Applications of Differential Equations 4

Logistic Growth: Most populations are limited by food, space, or waste build-up, thus, cannot continue to grow according to Malthusian growth

- The Logistic growth model has a Malthusian growth term and a term limiting growth due to crowding
- The differential equation is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right)$$

- P is the population, r is the Malthusian rate of growth, and M is the carrying capacity of the population
- We solve this problem later in the semester

Applications of Differential Equations 5

The van der Pol Oscillator: In electrical circuits, diodes show a rapid rise in current, leveling of the current, then a steep decline

- Biological applications include a similar approximation for nerve impulses
- The van der Pol Oscillator satisfies the differential equation

$$v'' + a(v^2 - 1)v' + v = b$$

- v is the voltage of the system, and a and b are constants

Applications of Differential Equations 6

Lotka-Volterra – Predator and Prey Model: Model for studying the dynamics of predator and prey interacting populations

- Model for the population dynamics when one predator species and one prey species are tightly interconnected in an ecosystem
- System of differential equations

$$\begin{aligned} x' &= a\,x - b\,xy \\ y' &= -c\,y + d\,xy \end{aligned}$$

- x is the prey species, and y is the predator species
- No explicit solution, but will study its behavior

Applications of Differential Equations 7

Forced Spring-Mass Problem with Damping: An extension of the spring-mass problem that includes viscous-damping caused by resistance to the motion and an external forcing function that is applied to the mass

- The model is given by

$$my'' + cy' + ky = F(t)$$

  - y is the position of the mass
  - m is the mass of the object
  - c is the damping coefficient
  - k is the spring constant
  - $F(t)$ is an externally applied force
- There are techniques for solving this

Applications of Differential Equations 8

Classification for Types of Differential Equations: Order of a Differential Equation

- The *order of a differential equation* is determined by the highest derivative in the differential equation
  - Harmonic oscillator, swinging-pendulum, van der Pol oscillator, and forced spring mass problem are 2^{nd} order differential equations
  - Malthusian and logistic growth and radioactive decay are 1^{st} order differential equations
  - Lotka-Volterra model is a 1^{st} order system of differential equations

Applications of Differential Equations 9

Classification for Types of Differential Equations: Linear and Nonlinear Differential Equations

- A differential equation is *linear* if the unknown dependent variable and its derivatives only appear in a linear manner
  - The Malthusian growth, radioactive decay, harmonic oscillator, and forced spring mass problem are linear differential equations
  - The swinging pendulum, van der Pol oscillator, logistic growth, and Lotka-Volterra model are nonlinear differential equations

Spring-Mass Problem 1

Spring-Mass Problem: Assume a mass attached to a spring without resistance satisfies the second order linear differential equation

$$y''(t) + 5y(t) = 0$$

Skip Example

Show that two of the solutions to this differential equation are given by

$$y_1(t) = 3\sin\left(\sqrt{5}t\right) \qquad \text{and} \qquad y_2(t) = 2\cos\left(\sqrt{5}t\right)$$

Spring-Mass Problem 2

Solution: Undamped spring-mass problem

- Take two derivatives of $y_1(t) = 3\sin\left(\sqrt{5}t\right)$

$$y_1'(t) = 3\sqrt{5}\cos\left(\sqrt{5}t\right) \quad \text{and} \quad y_1''(t) = -15\sin\left(\sqrt{5}t\right)$$

- Substituting into the differential equation

$$y_1'' + 5y_1 = -15\sin\left(\sqrt{5}t\right) + 5\left(3\sin\left(\sqrt{5}t\right)\right) = 0$$

- Take two derivatives of $y_2(t) = 2\cos\left(\sqrt{5}t\right)$

$$y_2'(t) = -2\sqrt{5}\sin\left(\sqrt{5}t\right) \quad \text{and} \quad y_2''(t) = -10\cos\left(\sqrt{5}t\right)$$

- Substituting into the differential equation

$$y_2'' + 5y_2 = -10\cos\left(\sqrt{5}t\right) + 5\left(2\cos\left(\sqrt{5}t\right)\right) = 0$$

Damped Spring-Mass Problem 1

Damped Spring-Mass Problem: Assume a mass attached to a spring with resistance satisfies the second order linear differential equation

$$y''(t) + 2y'(t) + 5y(t) = 0$$

Skip Example

Show that one solution to this differential equation is

$$y_1(t) = 2\,e^{-t}\sin(2t)$$

Damped Spring-Mass Problem 2

Solution: Damped spring-mass problem

- The 1^{st} derivative of $y_1(t) = 2\,e^{-t}\sin(2t)$

$$y_1'(t) = 2e^{-t}(2\cos(2t)) - 2e^{-t}\sin(2t) = 2e^{-t}(2\cos(2t) - \sin(2t))$$

- The 2^{nd} derivative of $y_1(t) = 2\,e^{-t}\sin(2t)$

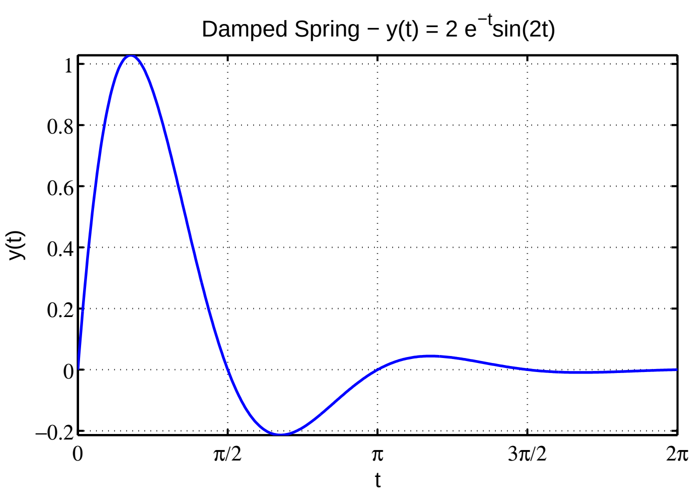
$$\begin{aligned} y_1''(t) &= 2e^{-t}(-4\sin(2t) - 2\cos(2t)) - 2e^{-t}(2\cos(2t) - \sin(2t)) \\ &= -2e^{-t}(4\cos(2t) + 3\sin(2t)) \end{aligned}$$

- Substitute into the spring-mass problem

$$\begin{aligned} y_1'' + 2y_1' + 5y &= -2e^{-t}(4\cos(2t) + 3\sin(2t)) \\ &\quad + 2(2e^{-t}(2\cos(2t) - \sin(2t))) + 5(2e^{-t}\sin(2t)) \\ &= 0 \end{aligned}$$

Damped Spring-Mass Problem 3

Graph of Damped Oscillator



Evaporation Example 1

Evaporation Example: Animals lose moisture proportional to their surface area

Skip Example

- If $V(t)$ is the volume of water in the animal, then the moisture loss satisfies the differential equation

$$\frac{dV}{dt} = -0.03\,V^{2/3}, \qquad V(0) = 8 \text{ cm}^3$$

- The initial amount of water is 8 cm^3 with t in days
- Verify the solution is

$$V(t) = (2 - 0.01t)^3$$

- Determine when the animal becomes totally dessicated according to this model
- Graph the solution

Evaporation Example 2

Solution: Show $V(t) = (2 - 0.01t)^3$ satisfies

$$\frac{dV}{dt} = -0.03\,V^{2/3}, \qquad V(0) = 8 \text{ cm}^3$$

- $V(0) = (2 - 0.01(0))^3 = 8$, so satisfies the initial condition
- Differentiate $V(t)$,

$$\frac{dV}{dt} = 3(2 - 0.01t)^2(-0.01) = -0.03(2 - 0.01t)^2$$

- But $V^{2/3}(t) = (2 - 0.01t)^2$, so

$$\frac{dV}{dt} = -0.03\,V^{2/3}$$

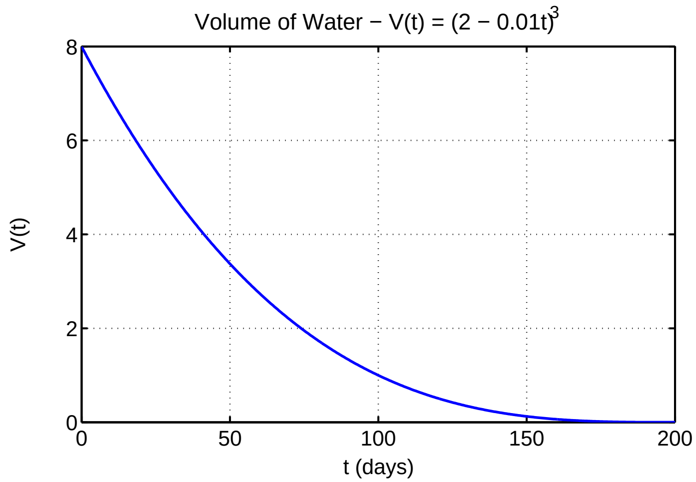
Evaporation Example 3

Solution (cont): Find the time of total dessication

- Must solve
$$V(t) = (2 - 0.01t)^3 = 0$$
- Thus,
$$2 - 0.01t = 0 \qquad \text{or} \qquad t = 200$$
- It takes 200 days for complete dessication

Evaporation Example 4

Graph of Dessication



Nonautonomous Example 1

Nonautonomous Example: Consider the nonautonomous differential equation with initial condition (**Initial Value Problem**):

$$\frac{dy}{dt} = -ty^2, \qquad y(0) = 2$$

- Show that the solution to this differential equation, including the initial condition, is
$$y(t) = \frac{2}{t^2 + 1}$$
- Graph of the solution

Nonautonomous Example

Solution: Consider the solution

$$y(t) = \frac{2}{t^2+1} = 2(t^2+1)^{-1}$$

- The initial condition is

$$y(0) = \frac{2}{0^2+1} = 2$$

- Differentiate $y(t)$,

$$\frac{dy}{dt} = -2(t^2+1)^{-2}(2t) = -4t(t^2+1)^{-2}$$

- However,

$$-ty^2 = -t(2(t^2+1)^{-1})^2 = -4t(t^2+1)^{-2}$$

- Thus, the differential equation is satisfied

Nonautonomous Example

Solution of Nonautonomous Differentiation Equation