Introduction

Managing More Integrals

- To date we have learned a collection of basic integrals
  - Polynomials
  - Power Law
  - Exponentials - $e^{kt}$
  - Trig Functions - $\sin(kt)$ and $\cos(kt)$
- Integration by substitution allows a substitution that reduces the integral to a simpler form
- This is basically this inverse of the Chain Rule of differentiation
- Apply to models using separable differential equations
  - The logistic growth model
  - Model for motion of an object subject to gravity

Logistic Growth Model for Yeast

Model considers a limited food source

- After a lag period, the organisms begin growing according to Malthusian growth
- As the food source becomes limiting, the growth of the organism slows and the population levels off
- This behavior is modeled by adding a negative quadratic term to the Malthusian growth model

$$\frac{dP}{dt} = rP \left( 1 - \frac{P}{M} \right) \quad \text{with} \quad P(0) = P_0$$
**Experiment:** G. F. Gause (*Struggle for Existence*) studied standard brewers yeast, *Saccharomyces cerevisiae*

- *S. cerevisiae* placed in a closed vessel, where nutrient was changed regularly (every 3 hours)
- Simulates a constant source of nutrient

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<td>12.77</td>
<td>12.87</td>
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*Model:* The *Logistic Growth Model* that best fits the data is

\[
\frac{dP}{dt} = 0.259 P \left( 1 - \frac{P}{12.7} \right), \quad \text{with} \quad P(0) = 1.23
\]

- How do we find the solution to this nonlinear differential equation?
- This is a separable equation
- The integral for *P* involves two integration techniques
- We’ll concentrate on the the integration by substitution

**Integration by Substitution**

- Integration is the inverse of differentiation
- Many functions that do not have an antiderivative
- **Integration by substitution** extends the number of integrable functions
- This technique is the inverse of the chain rule of differentiation
- The substitution technique finds a function that reduces an integral to an easier form
Example 1: Let \( a \) be a constant and consider the integral
\[
\int (x + a)^n \, dx
\]
Make the substitution \( u = x + a \), and the derivative gives the differentials \( du = dx \), so
\[
\int (x + a)^n \, dx = \int u^n \, du = \frac{u^{n+1}}{n+1} + C = \frac{(x + a)^{n+1}}{n+1} + C.
\]

Example 2: Consider the integral
\[
\int x e^{-x^2} \, dx
\]
Make the substitution \( u = -x^2 \), and the derivative gives the differentials \( du = -2x \, dx \), so
\[
\int x e^{-x^2} \, dx = \int e^{-x^2} \left( -\frac{1}{2} \right) (-2x) \, dx = \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{-x^2} + C.
\]

Example 3: Consider the integral
\[
\int (x^2 + 2x + 4)^3 (x + 1) \, dx
\]
Make the substitution \( u = x^2 + 2x + 4 \), and the derivative gives the differentials \( du = (2x + 2) \, dx \), so
\[
\int (x^2 + 2x + 4)^3 (x + 1) \, dx = \frac{1}{2} \int (x^2 + 2x + 4)^3 (2x + 2) \, dx = \frac{1}{2} \int u^3 \, du = \frac{u^4}{8} + C = \frac{(x^2 + 2x + 4)^4}{8} + C.
\]

Integration by Substitution: What makes a good substitution?

- Choose \( u \) such that when \( u \) and \( du \) are substituted for the expression of \( x \) under the integrand, the remaining integral became of one of the basic integrals solved earlier.
- There are a few choices that are very natural for a substitution.
  - Let \( u \) be any expression of \( x \) in the exponent of the exponential function \( e \) or the argument of any trigonometric functions or the logarithm function.
  - Let \( u \) be an expression of \( x \) inside parentheses raised to a power, where you should be able to see the derivative of that expression multiplying this expression to a power.
**Introduction**

Logistic Growth Model for Yeast

Integration by Substitution

Return to Logistic Growth

Examples

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**Return to Logistic Growth:** The Logistic Growth Model is

\[ \frac{dP}{dt} = rP \left(1 - \frac{P}{M}\right) = -rP \left(\frac{P}{M} - 1\right) \]

- Separate Variables to give
  
  \( \int \frac{dP}{P \left(\frac{P}{M} - 1\right)} = -r \int dt \)

  - The integral on the right is very easy to solve
  - The integral on the left requires a technique from algebra
    - Fraction is split into two simple fractions (reverse of a common denominator)
      
      \[ \frac{1}{P \left(\frac{P}{M} - 1\right)} = \frac{1}{\left(\frac{P}{M} - 1\right)} - \frac{1}{P} \]

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**Separated Differential Equation:** From fractional form above, write the integral as

\[ \int \frac{dP}{P \left(\frac{P}{M} - 1\right)} = \int \frac{dP}{\left(\frac{P}{M} - 1\right)} - \int \frac{dP}{P} \]

- One integral is easy
  
  \[ \int \frac{dP}{P} = \ln |P| + C \]

- For the other make the substitution \( u = \frac{P}{M} - 1 \), so \( du = \frac{dP}{M} \)

  \[ \frac{1}{M} \int \frac{dP}{\left(\frac{P}{M} - 1\right)} = \int \frac{du}{u} = \ln |u| = \ln \left|\frac{P}{M} - 1\right| \]

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**Separated Differential Equation:**

\[ \int \frac{dP}{P \left(\frac{P}{M} - 1\right)} = -r \int dt = -rt + C \]

- From results above
  
  \[ \ln \left|\frac{P}{M} - 1\right| - \ln |P| = -rt + C \]

- Thus,
  
  \[ \ln \left|\frac{\frac{P}{M} - 1}{P}\right| = \ln \left|\frac{P - M}{MP}\right| = -rt + C \]

- Exponentiating,
  
  \[ \left|\frac{P(t) - M}{MP(t)}\right| = e^{-rt+C} \]

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**Solution:** Removing the absolute value

\[ \frac{P(t) - M}{MP(t)} = Ae^{-rt} \]

- Solving for \( P(t) \) gives
  
  \[ P(t) = \frac{M}{1 - MAe^{-rt}} \]

- With the initial condition, \( P(0) = P_0 \)

  \[ P_0 = \frac{M}{1 - MA} \quad \text{or} \quad A = \frac{P_0 - M}{MP_0} \]

- Inserting this into the solution above gives

  \[ P(t) = \frac{P_0M}{P_0 + (M - P_0)e^{-rt}} \]

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Yeast Model: The best fitting yeast model
\[
\frac{dP}{dt} = 0.259 P \left( 1 - \frac{P}{12.7} \right), \quad \text{with} \quad P(0) = 1.23
\]

- The general logistic solution is
  \[
P(t) = \frac{P_0 M}{P_0 + (M - P_0) e^{-rt}}
\]
- It follows that
  \[
P(t) = \frac{15.62}{1.23 + 11.47 e^{-0.259 t}}
\]
- This function creates the standard *S-shaped curve* of logistic growth and has the *carrying capacity* of 12.7

Integration Example 1:
Consider the integral
\[
\int x^2 \cos(4 - x^3)\,dx
\]

**Solution:** A natural substitution is
\[
u = 4 - x^3 \quad \text{so} \quad du = -3x^2\,dx
\]
The solution of the integral is
\[
\int x^2 \cos(4 - x^3)\,dx = -\frac{1}{3} \int \cos(u)(-3x^2)\,du
\]
\[
= -\frac{1}{3} \sin(u) + C
\]
\[
= -\frac{1}{3} \sin(4 - x^3) + C
\]

Integration Example 2:
Consider the integral
\[
\int \left( \ln(2x) \right)^2 \frac{1}{x}\,dx
\]

**Solution:** A natural substitution is
\[
u = \ln(2x) \quad \text{so} \quad du = \frac{dx}{x}
\]
The solution of the integral is
\[
\int \left( \ln(2x) \right)^2 \frac{1}{x}\,dx = \int u^2\,du
\]
\[
= \frac{u^3}{3} + C
\]
\[
= \frac{1}{3} \left( \ln(2x) \right)^3 + C
\]
Differential Equation Example 1

Solution (cont): The integrations give
\[ \ln |y(t)| = \ln(t^2 + 4) + C \]

- Exponentiating
  \[ y(t) = e^{\ln(t^2+4)+C} = e^C(t^2 + 4) \]
- Note that \( e^C \) could be positive or negative depending on the initial condition
- From the initial condition, \( y(0) = 8 \), it follows that
  \[ y(t) = 2(t^2 + 4) \]

Differential Equation Example 2

Solution: Rewrite the differential equation
\[ \frac{dy}{dt} = 2t e^{t^2-y}, \quad y(0) = 2 \]

Separate the differential equation into the two integrals
\[ \int e^y dy = \int 2t e^{t^2} dt \]

Solution (cont): The right integral uses the substitution
\( u = t^2 \), so \( du = 2t dt \)
\[ \int e^y dy = e^y = \int 2t e^{t^2} dt = \int e^u du = e^u + C \]
- By substitution the implicit solution is
  \[ e^y = e^{t^2} + C \]
- Taking logarithms
  \[ y(t) = \ln \left( e^{t^2} + C \right) \]
- From the initial condition, \( y(0) = 2 = \ln(1 + C) \), it follows that
  \[ y(t) = \ln \left( e^{t^2} + e^2 - 1 \right) \]

Logistic Growth

Suppose that a population of animals satisfies the logistic growth equation
\[ \frac{dP}{dt} = 0.01 P \left( 1 - \frac{P}{2000} \right), \quad P(0) = 50 \]
- Find the general solution of this equation
- Determine how long it takes for this population to double
- Find how long it takes to reach half of the carrying capacity
Solution: We separate this logistic growth model

\[ \int \frac{dP}{P \left( \frac{P}{2000} - 1 \right)} = -0.01 \int dt = -0.01 t + C \]

- The **Fundamental Theorem Algebra** gives

\[ \frac{1}{P \left( \frac{P}{2000} - 1 \right)} = \frac{1}{2000} \left( \frac{P}{2000} - 1 \right) - \frac{1}{P} \]

- We use the substitution \( u = \frac{P}{2000} \), so \( du = \frac{P}{2000} \)

\[ \frac{1}{2000} \int \frac{dP}{\left( \frac{P}{2000} - 1 \right)} = \int \frac{du}{u} - \int \frac{dP}{P} = -0.01 t + C \]

Solution (cont): From the substitution \( u = \frac{P}{2000} \)

\[ \int \frac{du}{u} - \int \frac{dP}{P} = -0.01 t + C \]

- Thus,

\[ \ln |u| - \ln |P| = \ln \left| \frac{P - 2000}{2000} \right| - \ln |P| = -0.01 t + C \]

- So,

\[ \ln \left| \frac{P - 2000}{2000 P} \right| = -0.01 t + C \]

Solution (cont): Exponentiating the previous expression

\[ \frac{P(t) - 2000}{2000 P(t)} = e^{-0.01 t + C} = Ae^{-0.01 t} \]

- Solving for \( P(t) \),

\[ P(t) = \frac{2000}{1 - 2000 A e^{-0.01 t}} \]

- With the initial condition, \( P(0) = 50 \),

\[ P(t) = \frac{2000}{1 + 39 e^{-0.01 t}} \]

Solution (cont): The logistic growth model is

\[ P(t) = \frac{2000}{1 + 39 e^{-0.01 t}} \]

- The population doubles when

\[ P(t_d) = \frac{2000}{1 + 39 e^{-0.01 t_d}} = 100 \]

- Thus,

\[ 1 + 39 e^{-0.01 t_d} = 20 \quad \text{or} \quad e^{0.01 t_d} = \frac{39}{19} \]

- Solving for doubling time

\[ t_d = 100 \ln \left( \frac{39}{19} \right) = 71.9 \]
Logistic Growth

Solution (cont): The logistic growth model is

\[ P(t) = \frac{2000}{1 + 39 e^{-0.017t}} \]

The population reaches half the carrying capacity when

\[ P(t_h) = \frac{2000}{1 + 39 e^{-0.017 t_h}} = 1000 \]

Thus,

\[ 1 + 39 e^{-0.017 t_h} = 2 \quad \text{or} \quad e^{0.017 t_h} = 39 \]

Solving for doubling time

\[ t_h = 100 \ln(39) = 366.4 \]

Lake Pollution with Seasonal Flow

Lake Pollution with the Seasonal Flow: Often the flow rate into a lake varies with the season

- Suppose that a 200,000 m³ lake maintains a constant volume and is initially clean
- A river flowing into the lake has 6 µg/m³ of a pesticide
- Assume that the flow of the river has the sinusoidal form

\[ f(t) = 100(2 - \cos(0.0172 t)) \]

where \( t \) is in days

- Find and solve the differential equation describing the concentration of the pesticide in the lake
- Graph the solution for 2 years

Solution: Begin by creating the differential equation

- The change in the amount of pesticide, \( A(t) \), equals the amount entering - the amount leaving

\[ \frac{dA(t)}{dt} = 600(2 - \cos(0.0172 t)) - 100(2 - \cos(0.0172 t))c(t) \]

Concentration satisfies \( c(t) = \frac{A(t)}{200,000} \); so

\[ \frac{dc}{dt} = -\frac{(2 - \cos(0.0172 t))}{2000}(c - 6) \]

Separating variables

\[ \int \frac{dc}{c - 6} = -\frac{1}{2000} \int (2 - \cos(0.0172 t)) dt \]

Integrating

\[ \ln(u) = \ln(c(t) - 6) = -0.0005 \left( 2t - \frac{\sin(0.0172 t)}{0.0172} \right) + C \]

By exponentiating this implicit solution, using the initial condition \( c(0) = 0 \), and letting \( \frac{1}{0.0172} = 58.14 \), the solution becomes

\[ c(t) = 6 \left( 1 - e^{-0.0005(2t-58.14) \sin(0.0172 t)} \right) \]
Graph: Consider solution for 2 yr or 730 days

\[ c(t) = 6 \left( 1 - e^{-0.0005(2t-58.14 \sin(0.0172 t))} \right) \]