Calculus for the Life Sciences

Lecture Notes – Function Review

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Outline

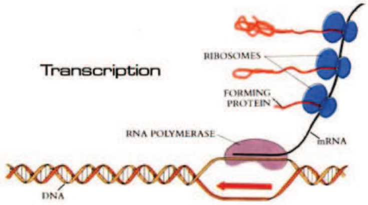
Rate of mRNA Synthesis

- DNA in *E. coli* provides the genetic code for all of the proteins
- DNA code used either for all aspects of the growth, maintenance, and reproduction of the cell
- The synthesis of proteins follows the processes of transcription and translation
- Proteins key for all cellular processes

Transcription

Transcription of a bacterial gene

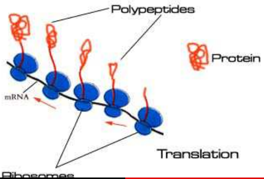
- A controlled sequence of steps, RNA polymerase, reads genetic code and produces a complementary messenger RNA (mRNA) template
- The mRNA is a short-lived blueprint for the production of a specific protein with a particular activity



Translation

Translation of a bacterial mRNA

- Begins shortly after transcription starts, with ribosomes reading the triplet codons on the mRNA
- Ribosome assembles a series of specific amino acids, forming a polypeptide
- Polypeptide probably folds passively into a tertiary structure which often combines with other proteins to become active or an enzyme



Rate of mRNA Synthesis

Rate of mRNA Synthesis

- The rate of growth of a bacterial cell depends on the rate at which it assembles all of its cellular components inside the cell
- The rate of production of different components inside the cell varies depending on the length of time it takes for a cell to double
- The table below shows the doublings/hr, μ, and the rate of mRNA synthesis (nucleotides/min/cell), $r_m \times 10^5$

μ	0.6	1.0	1.5	2.0	2.5
r_m	4.3	9.1	13	19	23

Linear Model for Rate of mRNA Synthesis

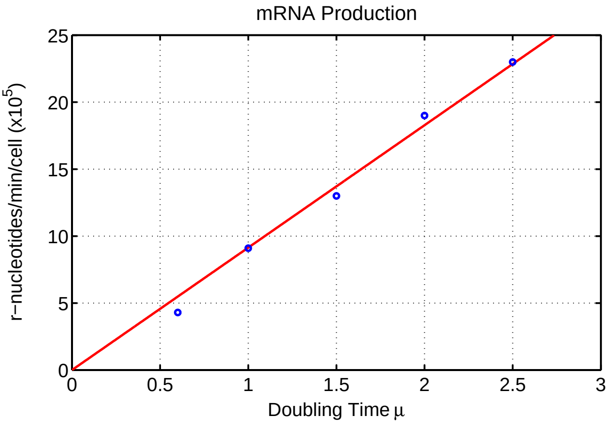
- Instability of the mRNA implies its rate of production closely approximates the rate of growth of a cell
- The data lie almost on a straight line passing through the origin
- Linear mathematical model of the form

$$r_m = a\mu$$

  for some value of a
- Want to find the best linear model by varying the slope, a

Graph of Data and Best Linear Model

Graph of Data and Best Linear Model



Least Squares Best Fit to Linear Model 1

Linear model passing through the origin has the form

$$r_m = a\mu$$

- The linear least squares best fit of this model to the data uses only the slope of the model, a
- The sum of the squares of the errors is computed from each of the error terms

$$\begin{aligned}
e_1^2 &= (4.3 - 0.6a)^2 \\
e_2^2 &= (9.1 - a)^2 \\
e_3^2 &= (13 - 1.5a)^2 \\
e_4^2 &= (19 - 2a)^2 \\
e_5^2 &= (23 - 2.5a)^2
\end{aligned}$$

Least Squares Best Fit to Linear Model 2

Sum of Square Errors is given by

$$J(a) = \sum_{i=1}^{5} e_i^2$$

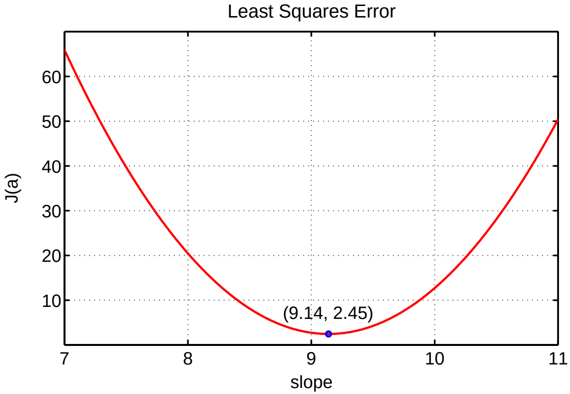
which reduces to

$$J(a) = 13.86\,a^2 - 253.36\,a + 1160.3$$

- $J(a)$ is a **quadratic function** representing the sum of the squares of the errors
- The best fit of the model is the smallest value of $J(a)$
- This occurs the vertex, a_v, of this quadratic equation

Graph of Least Squares Function $J(a)$

Graph of Least Squares Function – **Least Squares Best fit** when a is at a minimum, the vertex $a_v = 9.14$



Lambert-Beer Law 1

Concentration and Absorbance

- A spectrophotometer uses the Lambert-Beer law to determine the concentration of a sample (c) based on the absorbance of the sample (A)
- The ion dichromate forms an orange/yellow that has a maximum absorbance at 350 nm and is often used in oxidation/reduction reactions
- The **Lambert-Beer law** for the concentration of a sample from the absorbance satisfies the linear model

$$c = mA$$

where m is the slope of the line (assuming the spectrophotmeter is initially zeroed)

Lambert-Beer Law 2

Spectrophotometer data for an redox reaction

- Data collected on some known samples

A	0.12	0.32	0.50	0.665
c (mM)	0.05	0.14	0.21	0.30

- Determine the quadratic function $J(m)$ that measures the sum of the squares of the error of the linear model to the data
- Sketch a graph of $J(m)$ and find the vertex of this quadratic function
- Sketch a graph of the data and the line that best fits the data
- Use this model to determine the concentration of two unknown samples that have absorbances of $A = 0.45$ and 0.62

Lambert-Beer Law 3

Solution: Given the linear model $c = mA$, the sum of square errors satisfies

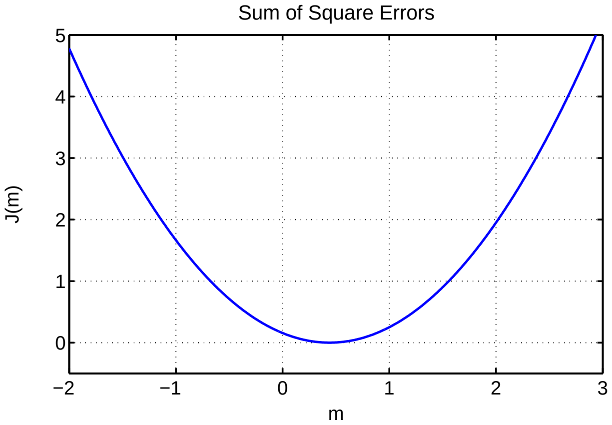
$$\begin{aligned} J(m) &= e_1^2 + e_2^2 + e_3^2 + e_4^2 \\ &= (0.05 - 0.12m)^2 + (0.14 - 0.32m)^2 + (0.21 - 0.50m)^2 + (0.30 - 0.66m)^2 \\ &= 0.8024m^2 - 0.7076m + 0.1562 \end{aligned}$$

The vertex has $m_v = \frac{0.7076}{2(0.8024)}$, so

$$(m_v, J(m_v)) = (0.44093, 0.00019995)$$

Lambert-Beer Law

Solution (cont): Graph of $J(m)$



Lambert-Beer Law 5

Solution (cont): Since the vertex has $m_v = 0.44093$, the **best linear model** is

$$c = 0.441\,A$$

For an absorbance $A = 0.45$

$$c(0.45) = 0.441(0.45) = 0.198$$

The best model predicts a concentration of 0.198 nM

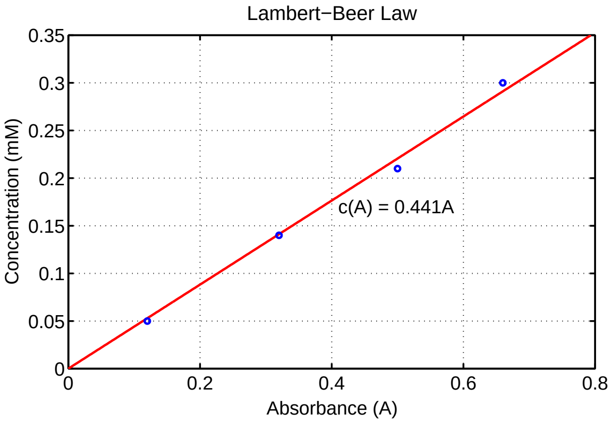
For an absorbance $A = 0.62$

$$c(0.62) = 0.441(0.62) = 0.273$$

The best model predicts a concentration of 0.273 nM

Lambert-Beer Law

Solution (cont): Graph of Best Linear Model and Data



Definitions and Properties of Functions

Definitions and Properties of Functions

- Functions form the basis for most of this course
- A function is a relationship between one set of objects and another set of objects with only one possible association in the second set for each member of the first set

Rate of mNA Synthesis Example

mRNA Example has two functions

- A set of possible cell doubling times, μ, to which was found a particular average rate of mRNA synthesis, r_m
- This subdivides into two functional representations
  - The experimental data, which represents a function with a finite set of points
  - The linear model, which creates a different function representing your theoretical expectations
- The sum of the squares of the errors between the data points and the model, $J(a)$, forms another function, where the set of possible slopes, a, in the model, each produced a number, $J(a)$, representing how far away the model was from the true data
  - Claim that the best model is when this function is at its lowest point

Definition of a Function

Definition: A **function** of a variable x is a rule f that assigns to each value of x a unique number $f(x)$. The variable x is the **independent variable**, and the set of values over which x may vary is called the **domain** of the function. The set of values $f(x)$ over the domain gives the **range** of the function

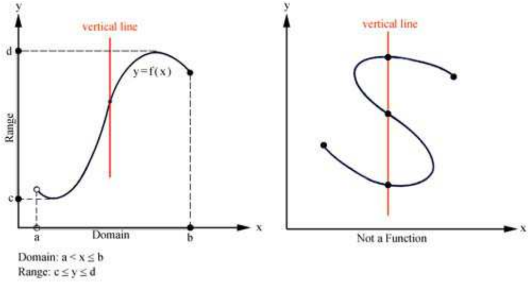
Definition of a Graph

Definition: The **graph of a function** is defined by the set of points (x, y) such that $y = f(x)$, where f is a function.

- Often a function is described by a **graph** in the xy-coordinate system
- By convention x is the domain of the function and y is the range of the function
- The **graph** is defined by the set of points $(x, f(x))$ for all x in the domain

Vertical Line Test

The **Vertical Line Test** states that a curve in the xy-plane is the graph of a function if and only if each vertical line touches the curve *at no more than one point*



Example of Domain and Range 1

Example 1: Consider the function

$$f(t) = t^2 - 1$$

Skip Example

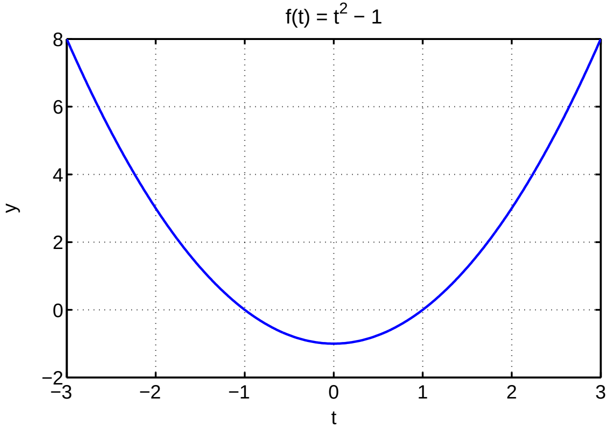
a. What is the range of $f(t)$ (assuming a domain of all t)?

Solution a: $f(t)$ is a parabola with its vertex at $(0, -1)$ pointing up.

Since the vertex is the low point of the function, it follows that range of $f(t)$ is $-1 \le y < \infty$

Graph of Example 1

Graph for the domain and range of $f(t)$



Example of Domain and Range

Example 1 (cont): More on the function

$$f(t) = t^2 - 1$$

b. Find the domain of $f(t)$, if the range of f is restricted to $f(t) < 0$

Solution b: Solving $f(t) = 0$ gives $t = \pm 1$

It follows that the domain is $-1 < t < 1$

Addition and Multiplication of Functions

Example 2: Let $f(x) = x - 1$ and $g(x) = x^2 + 2x - 3$

Skip Example

Determine $f(x) + g(x)$ and $f(x)g(x)$

Solution: The addition of the two functions

$$f(x) + g(x) = x - 1 + x^2 + 2x - 3 = x^2 + 3x - 4$$

The multiplication of the two functions

$$\begin{aligned} f(x)g(x) &= (x-1)(x^2+2x-3) \\ &= x^3 + 2x^2 - 3x - x^2 - 2x + 3 \\ &= x^3 + x^2 - 5x + 3 \end{aligned}$$

Addition of Function

Example 3: Let

$$f(x) = \frac{3}{x-6} \quad \text{and} \quad g(x) = -\frac{2}{x+2}$$

Skip Example

Determine $f(x) + g(x)$

Solution: The addition of the two functions

$$\begin{aligned} f(x) + g(x) &= \frac{3}{x-6} + \frac{-2}{x+2} = \frac{3(x+2) - 2(x-6)}{(x-6)(x+2)} \\ &= \frac{x+18}{x^2 - 4x - 12} \end{aligned}$$

Composition of Functions

Composition of Functions is another important operation for functions

Given functions $f(x)$ and $g(x)$, the composite $f(g(x))$ is formed by inserting $g(x)$ wherever x appears in $f(x)$

Note that the domain of the composite function is the range of $g(x)$

Composition of Functions

Example 4: Let

$$f(x) = 3x + 2 \quad \text{and} \quad g(x) = x^2 - 2x + 3$$

Skip Example

Determine $f(g(x))$ and $g(f(x))$

Solution: For the first composite function

$$f(g(x)) = 3(x^2 - 2x + 3) + 2 = 3x^2 - 6x + 11$$

The second composite function

$$g(f(x)) = (3x + 2)^2 - 2(3x + 2) + 3 = 9x^2 + 6x + 3$$

Clearly, $f(g(x)) \neq g(f(x))$

Even and Odd Functions

A function f is called:

1. **Even** if $f(x) = f(-x)$ for all x in the domain of f. In this case, the graph is symmetrical with respect to the y-axis

2. **Odd** if $f(x) = -f(-x)$ for all x in the domain of f. In this case, the graph is symmetrical with respect to the origin

Example of Even Function

Consider our previous example

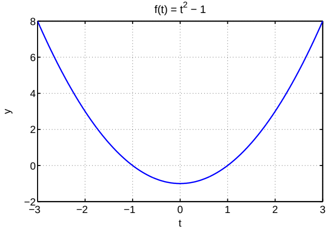
$$f(t) = t^2 - 1$$

Since

$$f(-t) = (-t)^2 - 1 = t^2 - 1 = f(t),$$

this is an even function.

The Graph of an Even Function is symmetric about the y-axis



One-to-One Function

Definition: A function f is **one-to-one** if whenever $x_1 \neq x_2$ in the domain, then $f(x_1) \neq f(x_2)$.

Equivalently, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Inverse Functions

Definition: If a function f is **one-to-one**, then its corresponding **inverse function**, denoted f^{-1}, satisfies:

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Since these are composite functions, the domains of f and f^{-1} are restricted to the ranges of f^{-1} and $f(x)$, respectively

Example of an Inverse Function 1

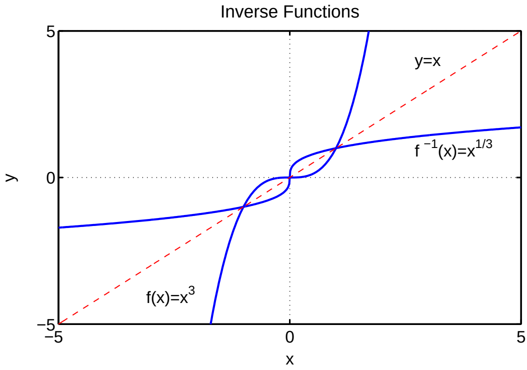
Consider the function
$$f(x) = x^3$$

It has the inverse function
$$f^{-1}(x) = x^{1/3}$$

The domain and range for these functions are all of x

$$f^{-1}(f(x)) = (x^3)^{1/3} = x = \left(x^{1/3}\right)^3 = f(f^{-1}(x))$$

Example of an Inverse Function 2



These functions are mirror images through the line $y = x$ (**the Identity Map**)