Discrete Logistic Growth Model

Qualitative Analysis of Logistic Growth Model

U. S. Population Models

Cobwebbing

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Introduction

Discrete Growth Models

- The Discrete Malthusian growth model shows exponential growth
- Most animal populations grow exponentially soon after settling
- With population growth, crowding pressure decreases the growth rate
  - Space and resource limitation
  - Toxic build up

Growing Culture of Yeast: Classic study by Carlson in 1913

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<th>Population</th>
<th>Time</th>
<th>Population</th>
<th>Time</th>
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<td>12</td>
<td>559.7</td>
<td>18</td>
<td>659.6</td>
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These data show a classic **S-shape curve**

Carlson (1913) Yeast data: Classic S-shape curve with initial accelerating growth, then eventually saturation

Discrete Dynamical Growth Model

There are two standard forms for discrete population models

One form uses a growth function, \( G(p_n) \)

\[ p_{n+1} = p_n + G(p_n) \]

The population at the next time interval \((n+1)\) equals the population at the current time interval \((n)\) plus the net growth of the current population, \(G(p_n)\)

Discrete Dynamical Model with Updating Function

A more general form satisfies

\[ p_{n+1} = F(p_n) \]

- An iterative map – the population at the \((n+1)^{st}\) generation depends on the population at the \(n^{th}\) generation
- The function \(F(p)\) is called the updating function
- The graph of the updating function
  - The \((n+1)^{st}\) generation is on the vertical axis
  - The \(n^{th}\) generation is on the horizontal axis
  - Usually want identity map to find equilibria

Logistic Growth Model

- Malthusian growth uses a linear updating function and grows exponentially without bound
- Most populations have a decreasing growth rate due to crowding effects
- This is the Logistic Growth model

\[ p_{n+1} = p_n + rp_n \left( 1 - \frac{p_n}{M} \right) \]

- This equation has the Malthusian growth model with the additional term \(-rp_n^2/M\)
- The parameter \(M\) is called the carrying capacity of the population
Behavior of the Logistic Growth Model

- The Logistic growth model shows complicated dynamics – shown by ecologist May (1974)
- There is no exact solution to this discrete dynamical system
- Given the Logistic Growth model
  \[ p_{n+1} = p_n + rp_n \left(1 - \frac{p_n}{M}\right) \]
  - There are equilibria at 0 and \( M \)
  - The parameter \( r \) has restricted values (\( r < 3 \)) with more complex behavior for higher values of \( r \)

Recall the logistic growth model has the form

\[ p_{n+1} = p_n + rp_n \left(1 - \frac{p_n}{M}\right) \]

The best fitting model to the yeast data is

\[ p_{n+1} = 1.5612p_n - 0.000861p_n^2 \]

The first two data values to obtain \( p_{n+1} \) and \( p_n \)

The graph of the data is fit with the best quadratic passing through the origin

Recall the logistic growth model has the form

\[ p_{n+1} = p_n + rp_n \left(1 - \frac{p_n}{M}\right) \]

Use successive data values to obtain \( p_{n+1} \) and \( p_n \)

The first two points are (9.6, 18.3) and (18.3, 29.0) with others found similarly

The graph of the data is fit with the best quadratic passing through the origin
Simulation: The model is easily simulated and by varying the initial population to $p_1 = 15.0$, a best fit to the data is found.

Consider the general discrete dynamical model:

$$p_{n+1} = F(p_n)$$

Study the qualitative behavior of discrete dynamical equations

- The first step in any analysis is finding equilibria
- This is simply an algebraic equation
- An equilibrium point of a discrete dynamical system is where there is no change in the variable from one iteration to the next
- Mathematically, $p_e = F(p_e)$
- Geometrically, this is when $F(p)$ crosses the identity map

Equilibria for Logistic Growth Model

Consider the logistic growth model:

$$p_{n+1} = p_n + rp_n \left(1 - \frac{p_n}{M}\right)$$

If $r > 0$, then equilibria satisfy

$$p_e = p_e + rp_e \left(1 - \frac{p_e}{M}\right)$$

$$rp_e \left(1 - \frac{p_e}{M}\right) = 0$$

Thus, $p_e = 0$ or $p_e = M$

The equilibria for the Logistic growth model are either

- The trivial solution $p_e = 0$ (no population) or
- The carrying capacity $p_e = M$

Simulation of the logistic growth model

- Simulation for $r = 0.5$ monotonically approaches carrying capacity
- Simulation for $r = 1.8$ oscillates and approaches carrying capacity
Simulation of the logistic growth model

- Simulation for $r = 2.3$ oscillates about equilibrium (Period 2)
- Simulation for $r = 2.65$ oscillates chaotically about equilibrium

**Stability of Logistic Growth Model**

Consider the **Trivial Equilibrium**, $p_e = 0$
- Since $f(p) = p + rp(1 - p/M)$, the derivative is
  $$f'(p) = 1 + r - \frac{2rp}{M}$$
- At $p_e = 0$, the derivative satisfies
  $$f'(0) = 1 + r$$
  - $r$ positive implies solutions growing away from $p_e = 0$
  - When the population is small, there are plenty of resources and the population grows (**exponentially**)
  - Near $p_e = 0$ solutions behave like **Malthusian growth**

Consider the **Carrying Capacity Equilibrium**, $p_e = M$
- Since the derivative is
  $$f'(p) = 1 + r - \frac{2rp}{M}$$
- At $p_e = M$, the derivative satisfies
  $$f'(M) = 1 - r$$
  - There are several possible behaviors of the solution near the carrying capacity equilibrium
If \( f'(p_e) > 1 \)
- Solutions of the discrete dynamical model grow away from the equilibrium (monotonically)
- The equilibrium is unstable

If \( 0 < f'(p_e) < 1 \)
- Solutions of the discrete dynamical model approach the equilibrium (monotonically)
- The equilibrium is stable

If \( -1 < f'(p_e) < 0 \)
- Solutions of the discrete dynamical model oscillate about the equilibrium and approach it
- The equilibrium is stable

If \( f'(p_e) < -1 \)
- Solutions of the discrete dynamical model oscillate about the equilibrium but move away from it
- The equilibrium is unstable

Behavior of Logistic Growth Model near \( p_e = M \)
- If \( 0 < r < 1 \), then the solution of the discrete logistic model monotonically approaches the equilibrium, \( p_e = M \), which was observed for the experiment with the yeast
- If \( 1 < r < 2 \), then the solution of the discrete logistic model oscillates about the equilibrium, \( p_e = M \), but the solution asymptotically approaches this equilibrium
- If \( 2 < r < 3 \), then the solution of the discrete logistic model oscillates about the equilibrium, \( p_e = M \), but the solution grows away from this equilibrium
- \( r > 3 \) results in negative solutions

Example of the Logistic Growth Model

**Example:** Consider the discrete logistic growth model

\[
p_{n+1} = F(p_n) = p_n + r p_n \left(1 - \frac{p_n}{1000}\right)
\]

We examine this model for \( r = 0.5, 1.5, \) and \( 2.3 \)
- Find all the equilibria for this model
- Determine the behavior of the solution near these equilibria
- Sketch a graph of the updating function and the identity map \( p_{n+1} = p_n \)
- Simulate the model, starting \( p_0 = 50 \) for 50 iterations

Solution: For the discrete logistic growth model

\[
p_{n+1} = F(p_n) = p_n + r p_n \left(1 - \frac{p_n}{1000}\right)
\]

the equilibria are \( p_e = 0 \) (extinction) and \( p_e = 1000 \) (carrying capacity)

**Stability Analysis** requires the derivative

\[
F'(p) = 1 + r - \frac{2rp}{1000}
\]

Since \( F'(0) = 1 + r > 1 \), it follows that \( p_e = 0 \) is an unstable equilibrium with solutions monotonically growing away
Example of the Logistic Growth Model

At the carrying capacity equilibrium, \( p_e = 1000 \) the derivative satisfies:

\[
F'(1000) = 1 + r - \frac{2r(1000)}{1000} = 1 - r
\]

- When \( r = 0.5 \), then \( F'(1000) = 0.5 \).
  - \( p_e = 1000 \) is a **stable** equilibrium with solutions **monotonically approaching** it
- When \( r = 1.5 \), then \( F'(1000) = -0.5 \).
  - \( p_e = 1000 \) is a **stable** equilibrium with solutions **oscillating** and **approaching** it
- When \( r = 2.3 \), then \( F'(1000) = -1.3 \).
  - \( p_e = 1000 \) is an **unstable** equilibrium with solutions **oscillating** and **moving away**

**Example of the Logistic Growth Model**

Simulation of the Logistic growth model

Graphing the updating function

U. S. Census with 3 Growth Models

**U. S. Census with 3 Models** - Models simulated with the census data from 1790 to 2010 and parameters fit

- Malthusian growth model

\[
P_{n+1} = 1.1460 \, P_n, \quad P_0 = 16.35
\]

- Nonautonomous growth model with \( n \) in decades after 1790

\[
P_{n+1} = (1.2960 - 0.009672 \, n) \, P_n, \quad P_0 = 6.307
\]

- Logistic growth model

\[
P_{n+1} = f(P_n) = P_n + 0.2244 \, P_n \left( 1 - \frac{P_n}{451.8} \right), \quad P_0 = 8.576
\]
Malthusian growth model

\[ P_{n+1} = (1 + r)P_n \quad \text{with} \quad P_0 \]

- Least squares best fit to census data

\[ P_n = P_0(1 + r)^n = 16.35(1.1460)^n \]

- The average growth over U.S. census history is \( r = 0.1460 \) per decade with best \( P_0 = 16.35 \) M
  - The sum of square errors is 2875.5
  - The \( P_0 \) is quite high and growth only matches growth near beginning of 20th century
  - Malthusian model isn’t expected to work well over long periods of time

Nonautonomous growth model

\[ P_{n+1} = (1 + k(t_n))P_n \quad \text{with} \quad P_0 \]

- Best least squares fit to census over U.S. history is

\[ P_{n+1} = (1.2960 + 0.009672n)P_n \quad \text{with} \quad P_0 = 6.307 \]

- This assumes a linear growth starting around 30% per decade and declining about 1% per decade
- Sum of square errors is 326.8 with \( P_0 \) being high
- This model matches the census quite well, but model difficult to analyze mathematically

Logistic growth model

\[ P_{n+1} = P_n + rP_n \left(1 - \frac{P_n}{M}\right) \quad \text{with} \quad P_0 \]

- Least squares best fit to census data

\[ P_{n+1} = P_n + 0.2244P_n \left(1 - \frac{P_n}{451.8}\right) \quad P_0 = 8.576 \]

- The growth rate is \( r = 0.2244 \), and carrying capacity is \( M = 451.8 \)
- The sum of square errors is 557.4 with \( P_0 \) being high
- This model matches the census data only slightly worse than the nonautonomous model, but easier to analyze

Graph of the 3 models and U.S. census data
**Logistic Updating Function**

- Direct fitting of the logistic time series to data can be numerically unstable
- Finding the quadratic updating function uses stable numerical routines
- Find \( P_n \) and \( P_{n+1} \) by taking successive pairs of census data
- This fit differs from direct time series with

\[
P_{n+1} = P_n + 0.2047 P_n \left( 1 - \frac{P_n}{532.6} \right)
\]

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**Graph of the Logistic Updating function**

Graph shows U. S. census data, quadratic for best updating, quadratic for best time series, and identity map

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**Summary: Future Projections**

- The Malthusian growth model is simple but simulates poorly for the entire history of the U. S.
- Nonautonomous growth model
  - Tracks poorly at first (poor \( P_0 \)), but only a few percent error over the last few decades
  - Model fails to account for high recent immigration and higher birth rates in immigrant communities
  - Model predicts population increases to a maximum near 450 M around 2100, then declines
- Logistic growth model
  - Tracks poorly at first (poor \( P_0 \)), but only a few percent error over the last few decades
  - Model predicts population increases to carrying capacity of 451.8 M, asymptotically (using smallest SSE)
Consider the **discrete dynamical model**

\[ p_{n+1} = f(p_n) \]

In the **Linear Discrete Dynamical Model section**, we showed a graphical method to view the local dynamics of this model called **cobwebbing**

Create a graph with the variable \( p_{n+1} \) on the vertical axis and \( p_n \) on the horizontal axis

Draw the graph of the **updating function**, \( f(p_n) \) and the **identity map**

\[ p_{n+1} = f(p_n) \quad \text{and} \quad p_{n+1} = p_n \]

Graphically, any intersection of the **updating function** and the **identity map**

\[ p_{n+1} = f(p_n) \quad \text{and} \quad p_{n+1} = p_n \]

produces an equilibrium

- The process of **cobwebbing** shows the dynamics of this discrete dynamical model
- Start at some point \( p_0 \) on the horizontal axis, then go vertically to \( f(p_0) \) to find \( p_1 \)
- Next go horizontally to the line \( p_{n+1} = p_n \)
- Go vertically to \( f(p_1) \) to find \( p_2 \)
- The process is repeated to give a geometric interpretation of the dynamics of the discrete model

**Cobwebbing – Breathing Model Example**

The model for a normal subject breathing an air mixture enriched with Ar satisfies the model

\[ c_{n+1} = (1 - q)c_n + q \gamma = 0.82 c_n + 0.0017 \]

Below reviews the cobwebbing process for this example

**Cobwebbing – Quadratic Example**

- Breathing model has a simple **linear updating function**
  - Unique equilibrium
  - Monotonic dynamics
- **Quadratic updating function** allows complicated dynamics
  - Logistic growth model is a quadratic dynamical model
  - Have observed monotonic, oscillatory, and chaotic dynamics
  - Show oscillatory dynamics for
    \[ p_{n+1} = 3p_n(1 - p_n) \]
    using a few steps of cobwebbing
- This example has equilibria at 0 and \( \frac{2}{3} \), the latter being between stable and unstable and oscillatory
Cobwebbing – Quadratic Example

\[ p_{n+1} = 3p_n (1-p_n) \]

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