Calculus for the Life Sciences

Lecture Notes – Discrete Malthusian Growth

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Spring 2017

Outline

United States Census

United States Census

- Constitution requires census every 10 years
- Census used for budgeting federal payments and representation in Congress
- Process can be politically charged
- Accurately predicting demographic data are important for planning communities in the future
- Calculations for the future populations uses sophisticated mathematical models
- Models are constantly improved and revised

Census Data

Census Data

1790	3,929,214	1870	39,818,449	1950	150,697,361
1800	5,308,483	1880	50,189,209	1960	179,323,175
1810	7,239,881	1890	62,947,714	1970	203,302,031
1820	9,638,453	1900	76,212,168	1980	226,545,805
1830	12,866,020	1910	92,228,496	1990	248,709,873
1840	17,069,453	1920	106,021,537	2000	281,421,906
1850	23,191,876	1930	122,775,046	2010	308,745,538
1860	31,443,321	1940	132,164,569		

Growth Rate of U. S.

Growth Rate in Early U. S.

The growth rate for the decade of 1790-1800

$$\frac{\text{Population in 1800}}{\text{Population in 1790}} = \frac{5,308,483}{3,292,214} = 1.351$$

The growth rate for the decade of 1800-1810

$$\frac{\text{Population in 1810}}{\text{Population in 1800}} = \frac{7,239,881}{5,308,483} = 1.364$$

The growth rate for the decade of 1810-1820

$$\frac{\text{Population in 1820}}{\text{Population in 1810}} = \frac{9,638,453}{7,239,881} = 1.331$$

Growth Rate of U. S.

- The growth rates for the decades following 1790, 1800, and 1810 are 35.1%, 36.4%, and 33.1%
- The average is 34.9% per decade
- This growth rate remains almost constant until 1860
- Suggests a **constant growth rate model**

Malthusian Growth Model

Malthusian Growth Model

- Simplest growth model uses a constant rate of growth, r
- Start with the population in 1790, P_0
- Population in the next decade is current population plus the population times the average growth rate

$$P_{n+1} = P_n + rP_n = (1+r)P_n$$

- Sequence of predicted populations based solely on population from preceding population

Malthusian Growth Model

Malthusian Growth Model for U. S. Population (early years)

Let $P_0 = 3,929,214$ (population 1790) and take $r = 0.349$

For 1800, model gives

$$P_1 = 1.349P_0 = 5,300,510$$

For 1810, model gives

$$P_2 = 1.349P_1 = 7,150,388$$

For 1820, model gives

$$P_3 = 1.349P_2 = 9,645,873$$

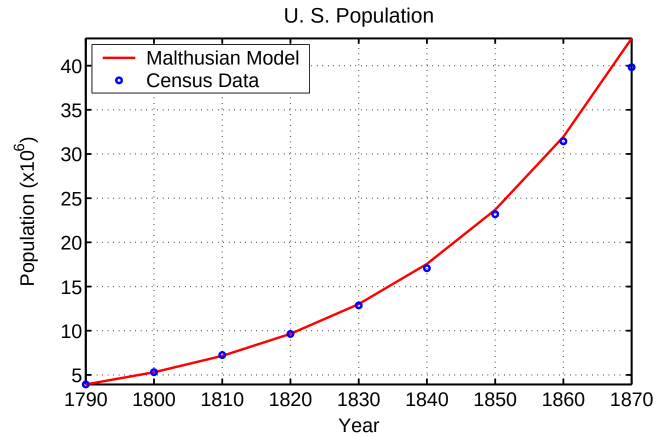
Malthusian Growth Model

Table for U. S. Population (early years)

Year	Census	Model $P_{n+1} = 1.349P_n$	% Error
1790	3,929,214	3,929,214	
1800	5,308,483	5,300,510	−0.15
1810	7,239,881	7,150,388	−1.24
1820	9,638,453	9,645,873	0.08
1830	12,866,020	13,012,282	1.14
1840	17,069,453	17,553,569	2.84
1850	23,191,876	23,679,765	2.10
1860	31,433,321	31,944,002	1.59
1870	39,818,449	43,092,459	8.22

Malthusian Growth Model

Graph of the **Malthusian Growth Model** and **Census Data** for the U. S.



Analysis of Growth Model

Early constant growth rate of about 35%

- Error remains small until 1870 because of the fairly constant rate of growth (Agrarian society)
- Most predicted populations are a little high, suggesting the 19^{th} century growth rate declined
- Civil War created dramatic decline in the growth rate
- More significantly, population demographics changed as the U. S. moved into the industrial revolution away from agriculture

Changing Growth Rate

Variation in Growth Rate

- Assume this Malthusian growth model were extended
  - In 1920, model predicts 192,365,343 (population in 1960s), which is 82% too high
  - In 1970, model predicts 859,382,645, which is 323% too high
- Average growth rate over census history is 22.3%
- Growth rate in 1920 is 15%, dropping to 13% in 1970
- Lowest growth rate during the Great Depression of 7.2%
- Latest growth rate for U. S. is 9.7%

Discrete Malthusian Growth

Discrete Malthusian Growth Model

$$P_{n+1} = P_n + rP_n = (1+r)P_n,$$

where r is the average growth rate

Next generation is proportional to the population of the current generation

- Named for Thomas Malthus (1766-1834)
- Example of **Discrete Dynamical system** or **Difference Equations**
- Population models using difference equations are common in ecological models

Solution of Discrete Malthusian Growth Model

The **Malthusian growth model** is one of few easily solved discrete models

$$P_1 = (1+r)P_0$$

$$P_2 = (1+r)P_1 = (1+r)^2 P_0$$

$$P_n = (1+r)P_{n-1} = ... = (1+r)^n P_0$$

General solution is given by

$$P_n = (1+r)^n P_0$$

Solution of Discrete Malthusian Growth Model

General solution of Malthusian growth model

$$P_n = (1+r)^n P_0$$

This solution shows why **Malthusian growth** is also known as **exponential growth**

The solution to the model is an exponential function with a base of $1+r$ and power n representing the number of iterations after the initial population

Example – Malthusian Growth 1

Suppose that a population of yeast, satisfying Malthusian growth, grows 10% in an hour. If the population begins with 100,000 yeast, then find the population at the end of 4 hours.

How long does it take for this population to double?

Skip Example

Solution: The Malthusian growth model is

$$P_{n+1} = (1+0.1)P_n, \qquad P_0 = 100,000$$

The general solution is

$$P_n = (1.1)^n P_0 = 100,000(1.1)^n$$

Example – Malthusian Growth 2

Solution (cont): The population after 4 hours

$$P_4 = 100,000(1.1)^4 = 146,410$$

For the solution to double

$$200,000 = 100,000(1.1)^n \quad \text{or} \quad (1.1)^n = 2$$

Taking logarithms

$$n\ln(1.1) = \ln(2) \quad \text{or} \quad n = \frac{\ln(2)}{\ln(1.1)} = 7.27 \text{ hr}$$

Example – Two Populations 1

Population Studies - Discrete Malthusian Growth

a. One species of insect grows according to the discrete Malthusian growth model

$$H_{n+1} = 1.06H_n, \qquad H_0 = 50,000$$

where n is in weeks

Find the population at the end of the first three weeks, H_1, H_2, and H_3

Determine how long it takes for this population to double

Skip Example

Example – Two Populations 2

Solution a: The Malthusian growth model satisfies

$$H_n = (1.06)^n H_0 = 50,000(1.06)^n$$

It follows that

$$H_1 = 50,000(1.06) = 53,000 \quad H_2 = 56,180 \quad H_3 = 59,551$$

The doubling time

$$2H_0 = (1.06)^n H_0$$

With logarithms

$$\ln(2) = n\ln(1.06) \quad \text{or} \quad n = \frac{\ln(2)}{\ln(1.06} = 11.90 \text{ weeks}$$

Example – Two Populations 3

b. Another insect species starts with a smaller population, but grows more quickly

$$G_{n+1} = 1.08G_n, \qquad G_0 = 10,000$$

Find the doubling time of this population of insects

Determine how long until the populations of the two species are equal

Solution b: This population satisfies

$$G_n = (1.08)^n G_0 = 10,000(1.08)^n$$

The doubling time satisfies

$$\ln(2) = n\ln(1.08) \quad \text{or} \quad n = \frac{\ln(2)}{\ln(1.08} = 9.0 \text{ weeks}$$

Example – Two Populations 4

Solution b (cont): The two populations are equal when

$$\begin{aligned}(1.08)^n G_0 &= (1.06)^n H_0 \\ 10,000(1.08)^n &= 50,000(1.06)^n \\ \left(\frac{1.08}{1.06}\right)^n &= 5 \\ n \ln\left(\frac{1.08}{1.06}\right) &= \ln(5) \\ n &= 86.1 \text{ weeks}\end{aligned}$$

The two populations are approximately equal after **86 weeks**

Compound Interest

Compound interest problems are closely related to Malthusian growth models

Start with an **initial principal** P_0 and an **annual interest rate** of r

The **principal** n years later, P_n satisfies

$$P_{n+1} = (1+r)P_n \qquad \text{given} \quad P_0$$

or

$$P_n = (1+r)^n P_0$$

Compound Interest - k times annually

When interest is compounded k times a year, the formula for the **amount of principal**, P_n, given an **initial principal** P_0 and an **annual interest rate** of r satisfies

$$P_n = \left(1 + \frac{r}{k}\right)^{kn} P_0$$

where n is in years

Example: Compound Interest 1

Example: Suppose you begin with $2,000 to invest. Bank A offers 6.25% interest compounded annually, while Bank B offers 6% interest compounded monthly. Which of these investments gives the better return?

Skip Example

Solution: Using the model above for Bank A, we have $k = 1$, $r = 0.0625$, and $P_0 = \$2,000$, so after a year

$$P_{1A} = (1 + 0.0625)^1(\$2,000) = \$2,125$$

Example: Compound Interest 2

Solution (cont): For Bank B, $k = 12$, $r = 0.06$, and P_0 is also \$2,000, so after one year

$$P_{1B} = \left(1 + \frac{0.06}{12}\right)^{12} (\$2,000) = \$2,123.36$$

So Bank A has a slightly better return by \$1.64

Annual Growth Rate 1

The population data for 1790 is 3,929,214, while the population data for 1800 is 5,308,483

This gives a decade growth rate of 35.1%

What is the annual growth rate?

Solution: If we let n be in years, then to find the annual growth rate, we solve

$$P_{10} = (1 + r)^{10} P_0$$

Annual Growth Rate 2

Solution (cont): Solve

$$\begin{aligned}
5,308,483 &= (1 + r)^{10} 3,929,214 \\
(1 + r)^{10} &= 1.351 \\
1 + r &= 1.351^{1/10} = 1.03054 \\
r &= 0.03054
\end{aligned}$$

It follows that annual growth rate is $r = 0.03054$ or 3.054%

Note that decade growth was 35.1%, which is more than 10 times the annual growth rate

This shows the effects of compounding interest

Example for Population Growth 1

Example The population in the U. S. was 203.3 million in 1970 and 226.5 million in 1980

Assume that the population is growing according to the discrete Malthusian growth model and find the annual growth rate of the population during this period of time

Use this information to project the population in 1990

The actual census gives the population in 1990 to be 248.7 million, so what is the percent error between the actual population and the modeling prediction?

Skip Example

Example for Population Growth

Solution: Let $P_0 = 203.3$ and $P_{10} = 226.5$

The decade growth rate satisfies:

$$\frac{P_{10}}{P_0} = \frac{226.5}{203.3} = 1.1141 = 1 + r_d$$

Thus, the growth rate per decade in 1970 was 11.41%

The annual growth rate satisfies:

$$\begin{aligned} 203.3(1+r_a)^{10} &= 226.5 \\ (1+r_a)^{10} &= 1.1141 \\ 1 + r_a &= 1.1141^{1/10} = 1.01086 \end{aligned}$$

The annual growth rate is $r_a = 0.01086$ or 1.086%

Example for Population Growth

Solution (cont): The discrete Malthusian growth model is

$$P_n = (1.01086)^n P_0 = 203.3(1.01086)^n$$

where n is in years

For $n = 20$ years in 1990, we obtain a population of

$$P_{20} = 203.3(1.01086)^{20} = 252.3 \text{ million}$$

The actual census is 248.7 million, so the percent error of this model is

$$100\left(\frac{252.3 - 248.7}{248.7}\right) = 1.45\%$$

Discrete Population Models

The general **Discrete Dynamical Population Model** (time-independent)

$$P_{n+1} = f(P_n)$$

This difference equation is **Autonomous**, since the function f depends only on the population

A more general **Discrete Dynamical Population Model** with **temporal** or **time dependence**

$$P_{n+1} = f(t_n, P_n)$$

This difference equation is **Nonautonomous**

Modeling U. S. Population

The average growth rate for U. S. over its history

$$r = 0.2233$$

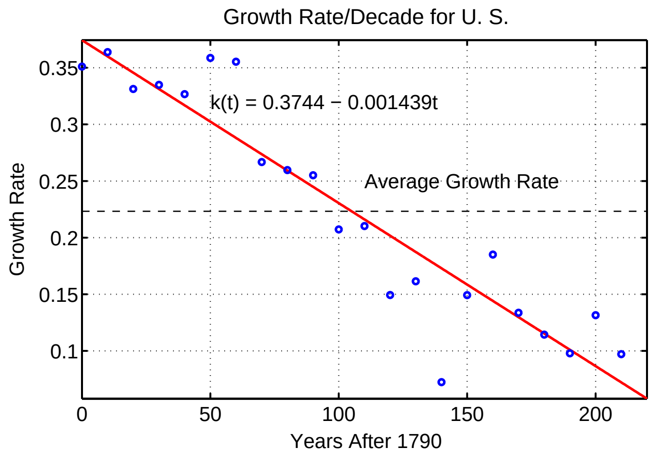
The best discrete Malthusian growth model is

$$P_{n+1} = 1.2233 P_n$$

This growth rate is too low for the early years, and too high for later years

Modeling U. S. Population

A **modified time dependent growth rate** is found by fitting a line through the data from 1790 to 1990



Modeling U. S. Population

The **best fit to the growth data** from 1790 to 1990 satisfies

$$k(t) = 0.3744 - 0.001439\,t$$

where t is the number of years after 1790

The **Nonautonomous Malthusian Growth Model** satisfies

$$P_{n+1} = (1 + k(t_n))P_n$$

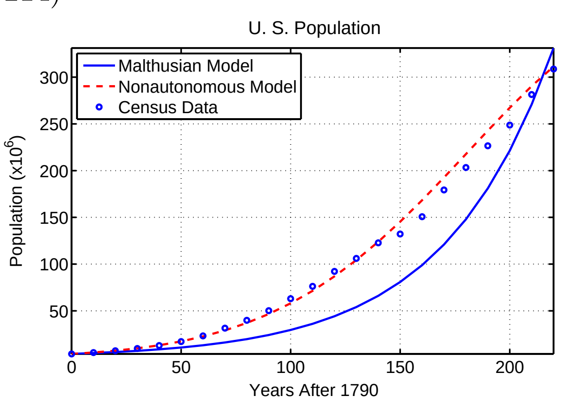
where $t_n = 10\,n$ and n is the number of decades after 1790

The model can be written

$$P_{n+1} = (1.3744 - 0.01439\,n)P_n$$

Modeling U. S. Population

Graph of the **Discrete Malthusian Growth Model** and **Nonautonomous Discrete Malthusian Growth Model** for the U. S. Population (with both models starting $P_0 = 3,929,214$)



Modeling U. S. Population

- The constant growth rate discrete Malthusian growth model does poorly over this long period of time
- The nonautonomous discrete Malthusian growth model does quite well for complete history
  - The average absolute percent error is only 5.1%
  - The maximum error occurs in 1950 with 11.7% error
- Both models over predict the 2010 census
  - The discrete Malthusian growth model predicts a population of 331,214,433
  - The nonautonomous discrete Malthusian growth model predicts a population of 311,407,591
  - These produce 7.3% and 0.9% errors, respectively

Example of Nonautonomous Growth 1

Example A population of arthropods is growing in a lake that begins to receive pesticide runoff from neighboring farm fields. The resulting pollution adversely affects the rate of growth of their population.

Suppose the nonautonomous Malthusian growth model for the arthropods is

$$A_{n+1} = (1 + k(t_n))A_n \qquad A_0 = 200(\text{per l}^3)$$

where n is weeks, $k(t_n) = 0.1 - 0.02n$, and A_n is the population density after n weeks

Skip Example

Population of the United States
Discrete Malthusian Growth
Compound Interest
Discrete Population Models
U. S. Population Modeling
Discrete Malthusian Growth
Varying Growth Rate

Example of Nonautonomous Growth 2

For the nonautonomous Malthusian growth model

$$A_{n+1} = (1.1 - 0.02\,n)A_n \qquad A_0 = 200$$

- Find the population at the end of the first three weeks, A_1, A_2, and A_3
- Find the maximum population density of these arthropods and when this occurs
- Determine when the lake becomes so polluted that the arthropod population dies out

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Example of Nonautonomous Growth 3

Solution: Unfortunately, this nonautonomous growth model does NOT have a general solution, like the Malthusian growth model above

The first three weeks, A_1, A_2, and A_3, are found by simulation

$$\begin{aligned} A_1 &= (1 + (0.1 - 0.02(0)))200 = (1.1)200 = 220, \\ A_2 &= (1 + (0.1 - 0.02(1)))220 = (1.08)220 = 237.6, \\ A_3 &= (1 + (0.1 - 0.02(2)))237.6 = (1.06)237.6 = 252.86. \end{aligned}$$

Finding when the maximum density occurs is easy, as it will occur when the growth rate falls to zero

$$k(t_n) = 0.1 - 0.02\,n = 0$$

which happens at $n_{max} = 5$

Population of the United States
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Example of Nonautonomous Growth 4

Solution (cont): Since there is no general solution, knowing when the maximum occurs only tells how far we need to simulate

Below are simulations for 10 weeks (which is easily done in Excel)

Week	Arthropods	Week	Arthropods
0	200	6	267
1	220	7	262
2	238	8	251
3	252	9	236
4	262	10	217
5	267		

Example of Nonautonomous Growth

Solution (cont): Theoretically, the arthropod population dies out when $1 + k(t_n) = 0$

$$1.1 - 0.02\,n = 0 \qquad \text{or} \qquad n = 55 \text{ weeks}$$

- Numerical simulations show that this population drops below 1 arthropod/l^3 in only 28 weeks
- From week 28 to 55, the population is very small
- Practically speaking, this population is extinct after the 28^{th} week
- There is some discrepancy between theoretical and numerical extinction with this more complicated model