Introduction

Differentiation of Trigonometric Functions

- Showed **Sine** and **Cosine** Models were good for periodic phenomenon
- Trigonometric functions are used to approximate more complicated behavior
- Joseph Fourier (1768-1830) used series of trigonometric functions to approximate other phenomena, such as harmonic motion of vibrating strings
- Tidal flow results from the interaction of differing gravitational fields
  - The complex dynamics are approximated by a short series of trigonometric functions with periods related to the astronomical bodies causing the tidal flow

Tides – Introduction

- Oceans are one of the great remaining frontiers
- Over half of the population on this planet lives within 100 miles of the oceans
- Tidal flow affects the daily dynamics of coastal zones
  - The Bay of Fundy in Newfoundland has tides rising over 16 meters in a 6.25 hour time period
  - In San Diego, the tidal flow is not so dramatic
- The tides do affect a variety of marine behaviors
- Most days there are two high tides (high-high and low-high) and two low tides (low-low and high-low)
How are tides predicted and what is the basis for their variability?

What causes the changes in amplitude and period between the high and low tides?

The primary forces generating the tides are from the gravity of the sun and the moon.

Gravitational forces of the Sun and Moon are the primary causes for generation of tides on Earth.

Four Dominant Tidal Forces

- Diurnal components about 24 hours (once per day)
  - $K_1$, the lunisolar force
  - $O_1$, the main lunar force

- Semidiurnal components about 12 hours (twice per day)
  - $M_2$, the main lunar force
  - $S_2$, the main solar force

- Periodic motion of the moon about the Earth (about 25 hours) cause variations

- When the moon, Earth, and sun align at either a full moon or a new moon, then the tides are at their highest and lowest as the forces of gravity enhance tidal flow

- The tides show the least variation when the moon is in its first or last quarter

- Other complications include
  - The elliptical orbit of the moon around the Earth
  - The elliptical orbit of the Earth around the sun
  - The influences of other planets
Mathematical Model for Tides

- What mathematical tools can help predict the tides?
- Use a series of trigonometric functions to approximate the behavior of the tides
- Standard programs use 12-14 trigonometric functions
- Next Slide are graphs of the high and low tides for San Diego for September 2002
  - The model is generated using only four trigonometric functions from the four forces described above
  - The data points indicate the actual values of the high and low tides from standard tide tables

First 2 weeks of Tides for San Diego in September 2002
Model and Data

Model for Height of Tides, \( h(t) \) in feet with \( t \) hours from midnight 1st day of the month

- The function \( h(t) \) is formed by the sum of four cosine functions and a constant
- The periods of the cosine functions reflect the periodic nature of the forces
  - \( K_1 \), lunisolar diurnal force with period \( p_1 = 23.934 \)
  - \( O_1 \), main lunar diurnal force with period \( p_2 = 25.819 \)
  - \( M_2 \), main lunar semidiurnal force with period \( p_3 = 12.421 \)
  - \( S_2 \), main solar semidiurnal force with period \( p_4 = 12.00 \)
- Periods are fixed based on the rotations of the moon and Earth
Mathematical Model for Tides

Model for Height of Tides

- The amplitudes associated with each force are $a_i, i = 1, ..., 4$
- The phase shifts associated with each force are $\phi_i, i = 1, ..., 4$
- A vertical shift satisfies $a_0$
- The parameters, $a_i$ and $\phi_i$, are fit using a least squares best fit to the high and low tides for the month of September 2002

$$h(t) = a_0 + \sum_{i=1}^{4} a_i \cos \left( \frac{2\pi}{p_i} (t - \phi_i) \right)$$

Best Fitting Parameters

<table>
<thead>
<tr>
<th>Force</th>
<th>Amplitude</th>
<th>Phase Shift</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>$a_1 = 0.878$</td>
<td>$\phi_1 = 16.246$</td>
</tr>
<tr>
<td>$O_1$</td>
<td>$a_2 = 0.762$</td>
<td>$\phi_2 = 14.311$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$a_3 = 1.993$</td>
<td>$\phi_3 = 6.164$</td>
</tr>
<tr>
<td>$S_2$</td>
<td>$a_4 = 0.899$</td>
<td>$\phi_4 = 10.857$</td>
</tr>
</tbody>
</table>

Model and Forces

- The strongest force affecting the tides is the semidiurnal main lunar force
- The highest and lowest tides of the month coincide with the new moon and full moon

<table>
<thead>
<tr>
<th>New Moon</th>
<th>First Quarter</th>
<th>Full Moon</th>
<th>Last Quarter</th>
</tr>
</thead>
<tbody>
<tr>
<td>September 6</td>
<td>September 13</td>
<td>September 21</td>
<td>September 29</td>
</tr>
</tbody>
</table>

Modelling Low and High Tides

- When do the highest and lowest tides occur based on the mathematical model?
- The high and low points of a function are the maxima and minima
- This uses differentiation of our model, $h(t)$
- The high and low tides should occur when $h'(t) = 0$
Differentiation of Sine and Cosine

- The derivative of these functions is found using the definition of the derivative and some trigonometric identities.

  **Derivative of Sine**
  \[
  \frac{d}{dx} \sin(x) = \cos(x)
  \]

  **Derivative of Cosine**
  \[
  \frac{d}{dx} \cos(x) = -\sin(x)
  \]

### More Generally, the Differentiation of Sine and Cosine

- **Derivative of Sine**
  \[
  \frac{d}{dt} \sin(\omega(t - \phi)) = \omega \cos(\omega(t - \phi))
  \]

- **Derivative of Cosine**
  \[
  \frac{d}{dt} \cos(\omega(t - \phi)) = -\omega \sin(\omega(t - \phi))
  \]

- Soon learn the **chain rule** of differentiation for more general cases.
Consider an example of **Two Sine Functions**

\[ f(t) = 2 \sin(2t) - 4 \sin(3t) \]

- Properties of trig function give for \( 2 \sin(2t) \):
  - Period: \( T_1 = \pi \)
  - Maxima occur: \( t_{\text{max}} = \frac{\pi}{4} + n\pi, \ n = -1, 0, 1, \ldots \)
  - Minima occur: \( t_{\text{min}} = \frac{3\pi}{4} + n\pi, \ n = -1, 0, 1, \ldots \)
- Properties of trig function give for \( -4 \sin(3t) \):
  - Period: \( T_2 = \frac{2\pi}{3} \)
  - Maxima occur: \( t_{\text{max}} = \frac{\pi}{3} + \frac{2n\pi}{3}, \ n = -1, 0, 1, \ldots \)
  - Minima occur: \( t_{\text{min}} = \frac{2\pi}{3} + \frac{2n\pi}{3}, \ n = -1, 0, 1, \ldots \)

**Common Period:** Information gives a period for \( f(t) \) of \( T = 2\pi \)

For the example of **Two Sine Functions**

\[ f(t) = 2 \sin(2t) - 4 \sin(3t), \]

the derivative is easily computed

\[ f'(t) = 4 \cos(2t) - 12 \cos(3t). \]

- Find **Maxima** and **Minima** by solving \( f'(t) = 0 \) or \( 4 \cos(2t) = 12 \cos(3t) \)
- This equation is NOT readily solved
- Need **numerical methods** to find **Extrema**

The graph of the **Two Sine Functions** example

\[ f(t) = 2 \sin(2t) - 4 \sin(3t), \]

- Graph shows the periodicity, \( T = 2\pi \)
- Each period has 3 **Relative Maxima** and **Minima**
  - For \( t \in [0, 2\pi] \), numerical **Relative Maxima** are \( (1.4604, 4.2206) \ (3.7117, 5.7786) \ (5.8283, 2.3366) \)
  - For \( t \in [0, 2\pi] \), numerical **Relative Minima** are \( (0.4549, -2.3366) \ (2.5715, -5.7786) \ (4.8228, -4.2206) \)
- For \( t \in [0, 2\pi] \), the **Absolute Maximum** and **Minimum** are \( (3.7117, 5.7786) \) and \( (2.5715, -5.7786) \)
- These **Critical Points** are readily found with **Maple**
High and Low Tides

- Model and data show the highest and lowest tides in the 1st and 3rd weeks (New and Full moon phases)
- Use the model to predict the highest high-high tide and lowest low-low tide for the first week
- Determine the error between the model and the actual values for these tides
- The times and heights of the high and low tides use the local extrema

Model for Tides

\[ h(t) = a_0 + \sum_{i=1}^{4} a_i \cos \left( \frac{2\pi}{p_i} (t - \phi_i) \right) \]

The derivative satisfies:

\[ h'(t) = -\sum_{i=1}^{4} \left( \frac{2\pi a_i}{p_i} \right) \sin \left( \frac{2\pi}{p_i} (t - \phi_i) \right) \]

- Solving \( h'(t) = 0 \) is too complicated to find the extrema by hand
- Easily solved by computers, such as Excel’s Solver or Maple

New Moon  There was a New moon on September 6, 2002

- The graphs show many local extrema for the month of September
- Localize the search for the extrema using the visual information from the graph
- In the first week, the data show that the highest tide is 6.7 ft on Sept. 6, while the lowest tide is -1.0 ft on the same day
- So what does our model using four cosine functions predict to be the highest and lowest tides of this week?

Low Tide Prediction for September 6, 2002

- Parameters were fit for complete month of September and given earlier
- Set \( h'(t) = 0 \) and solved with a computer
- Low Tide Prediction
  - The lowest tide of the first week from the model occurs when \( t_{min} = 124.58 \) hrs with a \( h(t_{min}) = -0.86 \) ft
  - This corresponds to Sept. 6 at 4:35 am
  - The actual low-low tide on Sept. 6 is -1.0 ft occurring at 3:35 am
  - The model overshoots the tide height by about 0.14 feet and misses the time by 60 minutes
High and Low Tides

High Tide Prediction for September 6, 2002

- Repeat process for High Tide
- High Tide Prediction
  - The highest tide of the first week from the model occurs when \( t_{\text{max}} = 142.56 \) hrs with a \( h(t_{\text{max}}) = 6.40 \) ft
  - This corresponds to Sept. 6 at 10:33 pm
  - The actual high-high tide on Sept. 6 is 6.7 ft occurring at 9:36 pm
  - The model undershoots the tide height by about 0.3 feet and misses the time by 57 minutes

Summary of Tide Model for September 2002

- The calculations above show that our model introduces a moderate error
- Model only uses four cosine functions to try to predict an entire month of high and low tides
- This is a reasonable approach to the problem
- Obviously, the addition of more trigonometric functions and more parameters can produce a much more accurate model
- Actual models use 12-14 trigonometric functions to model tides
- The information line (619-221-8824) for the San Diego Beach report gives tide information

Maximum Change in Temperature

The sine function can be used to approximate the temperature during a day

\[
T(t) = A + B \sin(\omega(t - \phi)),
\]

with constants \( A, B \geq 0, \omega > 0, \) and \( \phi \in [0, 24) \) are determined from the data

- Suppose that the coolest temperature for a day occurs at 3 am and is 56°F
- Assume at 3 pm, the hottest temperature of 82°F occurs
Maximum Change in Temperature

The temperature is modeled by

\[ T(t) = A + B \sin(\omega(t - \phi)), \]

- Find the constants that best fit the data for the temperature during the day assuming that the temperature has a 24 hour period
- Determine the times during the day that the temperature is rising most rapidly and falling most rapidly
- Give the rate of change of temperature at those times

Solution: The temperature during a day

\[ T(t) = A + B \sin(\omega(t - \phi)) \]

- The average temperature is
  \[ A = \frac{56 + 82}{2} = 69^\circ F \]
- The amplitude of this function is found from the difference between the high temperature and the average temperature
  \[ B = 82 - 69 = 13^\circ F \]
- The 24 hour periodicity gives
  \[ 24\omega = 2\pi \text{ or } \omega = \frac{\pi}{12} \]

Solution (cont): The temperature during a day

\[ T(t) = 69 + 13 \sin\left(\frac{\pi}{12}(t - \phi)\right) \]

- The maximum occurs at 3 pm or \( t = 15 \)
- The maximum of the sine function occurs at \( \frac{\pi}{2} \)
- The phase shift, \( \phi \), solves
  \[ \frac{\pi}{12}(15 - \phi) = \frac{\pi}{2} \]
- It follows that
  \[ 15 - \phi = 6 \text{ or } \phi = 9 \text{ hr} \]
**Solution (cont):** Graph shows temperature is rising most rapidly in the morning and falling most rapidly in the evening.

Model is

\[ T(t) = 69 + 13 \sin \left( \frac{\pi}{12} (t - 9) \right) \]

The derivative satisfies

\[ T'(t) = \frac{13\pi}{12} \cos \left( \frac{\pi}{12} (t - 9) \right) \]

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**Alternate Solution:** The maximum and minimum rate of change occurs when second derivative is zero

\[ T''(t) = -\frac{13\pi^2}{144} \sin \left( \frac{\pi}{12} (t - 9) \right) \]

- The sine function is zero when the argument is \( n\pi \)
- Solve \( \frac{\pi}{12} (t - 9) = n\pi, \quad n = 0, 1, ... \)
- Thus, \( t = 9 + 12n \quad n = 0, 1, ... \)
- This gives same result as before