Chain Rule

- Functional relationships where one measurable quantity depends on another, while the second quantity is a function of a third quantity
- This functional relationship is a composite function
- The differentiation of a composite function requires the chain rule

Average Height and Weight of Girls

- Over a range of ages the rate of growth of girls in height is constant
- Height and age are approximated well by a linear function
- Height and weight of animals satisfies an allometric model
Average Height and Weight of Girls

<table>
<thead>
<tr>
<th>age</th>
<th>height</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(years)</td>
<td>(cm)</td>
</tr>
<tr>
<td>1</td>
<td>75</td>
<td>9.5</td>
</tr>
<tr>
<td>2</td>
<td>87</td>
<td>11.8</td>
</tr>
<tr>
<td>3</td>
<td>94</td>
<td>15.0</td>
</tr>
<tr>
<td>4</td>
<td>102</td>
<td>15.9</td>
</tr>
<tr>
<td>5</td>
<td>108</td>
<td>18.2</td>
</tr>
<tr>
<td>6</td>
<td>114</td>
<td>20.0</td>
</tr>
<tr>
<td>7</td>
<td>121</td>
<td>21.8</td>
</tr>
</tbody>
</table>

Average Height and Weight of American Girls

Least Squares Best Fit: Model of Height as a function of age

\[ h(a) = 6.45a + 73.9 \]

Model shows that the average girl grows about 6.45 cm/yr

Composite Model: The linear model shows that the average girl grows about 6.45 cm/yr

- How do we find the rate of change in weight for a girl at any particular age (between 1 and 13)?
  - The Allometric Model gives the weight as a function of height
  - Create a composite function of the allometric model and the linear model to give a function of the weight as a function of age
  - The chain rule gives the rate of change of weight with respect to age
Chain Rule: Consider the composite function \( f(g(x)) \)

- Suppose that both \( f(u) \) and \( u = g(x) \) are differentiable functions
- The chain rule for differentiation of this composite function is given by
  \[
  \frac{df}{dx} = \frac{df}{du} \frac{du}{dx}
  \]
- Alternately, the chain rule is written
  \[
  \frac{d}{dx} (f(g(x))) = f'(g(x))g'(x)
  \]

Example – Chain Rule

Example - Chain Rule: Consider the the function \( h(x) = (x^2 + 2x - 5)^5 \)

Find \( h'(x) \)

Solution: Consider the composite of the functions

\( f(u) = u^5 \) and \( g(x) = x^2 + 2x - 5 \)

The derivatives of both \( f \) and \( g \) are

\( f'(u) = 5u^4 \) and \( g'(x) = 2x + 2 \)

From the chain rule

\[
\begin{align*}
  h'(x) &= 5(g(x))^4(2x + 2) \\
  h'(x) &= 5(x^2 + 2x - 5)^4(2x + 2)
\end{align*}
\]

Example 2 – Chain Rule

Example 2 - Chain Rule: Consider the function \( h(x) = e^{2-x^2} \)

Find \( h'(x) \)

Solution: Consider the composite of the functions

\( f(u) = e^u \) and \( g(x) = 2 - x^2 \)

The derivatives of both \( f \) and \( g \) are

\( f'(u) = e^u \) and \( g'(x) = -2x \)

From the chain rule

\[
  h'(x) = e^{2-x^2}(-2x)
\]

Chain Rule for Special Functions

- General Derivative of Exponential Function
  \[
  \frac{d}{dx} e^{f(x)} = e^{f(x)}f'(x)
  \]

- General Derivative of Logarithm Function
  \[
  \frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}
  \]

- General Derivative of Sine
  \[
  \frac{d}{dx} \sin(f(x)) = f'(x)\cos(f(x))
  \]

- General Derivative of Cosine
  \[
  \frac{d}{dx} \cos(f(x)) = -f'(x)\sin(f(x))
  \]
Example 3: Consider the function

\[ f(x) = e^{-3x} \cos(x^2 + 4) \]

Find the derivative of \( f(x) \)

**Solution:** This derivative uses the product and chain rule

\[
\begin{align*}
  f'(x) &= e^{-3x}(-2x \sin(x^2 + 4)) + \cos(x^2 + 4)(e^{-3x}(-3)) \\
  &= -e^{-3x}(2x \sin(x^2 + 4) + 3 \cos(x^2 + 4))
\end{align*}
\]

Example 4: Consider the function

\[ f(x) = 3x^2 \sin(\ln(x + 2)) \]

Find the derivative of \( f(x) \)

**Solution:** This derivative uses the product and chain rule

\[
\begin{align*}
  f'(x) &= (3x^2) \left( \frac{d}{dx} \sin(\ln(x + 2)) \right) + 6x \sin(\ln(x + 2)) \\
  &= 3x^2 \cos(\ln(x + 2)) \frac{1}{x + 2} + 6x \sin(\ln(x + 2))
\end{align*}
\]

Example 5: Consider the function

\[ f(x) = 4e^{-\cos(2x+1)} \]

Find the derivative of \( f(x) \)

**Solution:** This derivative uses the chain rule

\[ f'(x) = 4e^{-\cos(2x+1)}(2 \sin(2x + 1)) \]

**Rate of Change in Weight:** The example for the weight and height of a child given above found

- The weight \( W \) as a function of height \( h \) is
  \[ W(h) = 0.000720 \, h^{2.17} \]

- The height as a function of age is
  \[ h(a) = 6.45 \, a + 73.9 \]

- The composite function weight as a function of age
  \[ W(a) = 0.000720(6.45 \, a + 73.9)^{2.17} \]
**Composite Function:** Weight as a function of age

\[ W(a) = 0.000720(6.45a + 73.9)^{2.17} \]

From the chain rule, the derivative of the weight function is

\[ \frac{dW}{da} = \frac{dW}{dh} \cdot \frac{dh}{da} \]

\[ \frac{dW}{dh} = 2.17(0.000720)^{1.17} \quad \text{and} \quad \frac{dh}{da} = 6.45 \]

Combining these and substituting the expression for \( h \)

\[ W'(a) = 0.01008(6.45a + 73.9)^{1.17} \]

This graph is almost linear, since it is to the 1.17 power.

- The actual average weight changes are given for the data above.
- We see that the model under predicts the weight gain for older girls.
**Normal Distribution:** This is an important function in statistics

- Gives the classic **Bell curve**
- The **normal distribution function** is

\[ N(x) = \frac{a}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

- \(a\) is the normalizing factor
- \(\sigma\) is the **standard deviation**
- \(\mu\) is the **mean** of the distribution

**Solution:** Consider

\[ N(x) = \frac{a}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

The derivative is

\[
\frac{dN}{dx} = \frac{a}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left( -\frac{2(x-\mu)}{2\sigma^2} \right)
\]

The derivative is zero when \(x = \mu\), so there is a maximum at \((\mu, \frac{a}{\sigma})\)

\[ N(x) = \frac{a}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

- Find the maximum and points of inflection
- Plot this function for several values of \(\sigma\)
- Discuss the importance of the results

**Solution:** The derivative is

\[ \frac{dN}{dx} = -\frac{a(x-\mu)}{\sigma^3} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

The second derivative is

\[
\frac{d^2N}{dx^2} = \frac{a}{\sigma^3} \left( (x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left( -\frac{2(x-\mu)}{2\sigma^2} \right) + e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot 1 \right)
\]

The points of inflection occur at \(x = \mu \pm \sigma\) with

\[ N(\mu \pm \sigma) = \frac{a}{\sigma} e^{-\frac{1}{2}} \]
**Normal Distribution**

**Solution:** Graph of the Normal Distribution with \( \mu = 0 \) and \( \sigma = 1, 2, 3, 4 \)

![Graph of the Normal Distribution](image)

**Properties of Normal Distribution**

\[
N(x) = \frac{a}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
\]

- As noted above, the **mean** of the normal distribution is \( \mu \)
- The normal distribution is a **bell-shaped** curve centered about its mean
- The points of inflection occur one standard deviation, \( \sigma \), from the mean, \( \mu \)
- It can be shown that 68% of the area under the normal distribution occurs in the interval, \([-\sigma, \sigma]\)
- The **area** under a distribution function is important in measuring probabilities and confidence intervals for statistics

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**Hassell’s Model**

**Hassell’s Model** is often used in the study of insect populations

Consider **Hassell’s Updating function**:

\[
H(P) = \frac{81P}{(1 + 0.002P)^4}
\]

- Find the intercepts and any asymptotes for \( P \geq 0 \)
- Find the derivative of \( H(P) \) and determine all extrema
- Graph \( H(P) \) for \( P \geq 0 \)

**Solution:** Hassell’s model satisfies:

\[
H(P) = \frac{81P}{(1 + 0.002P)^4}
\]

- For \( H(P) \), the only intercept is \((0, 0)\)
- The power of \( P \) in the denominator (4) exceeds the power in the numerator (1), so there is a horizontal asymptote with \( H = 0 \)
Solution (cont): Next we find the derivative of Hassell’s model:

\[ H(P) = \frac{81P}{(1 + 0.002P)^4} \]

\[ \frac{dH}{dP} = \frac{81(1 + 0.002P)^4 \cdot 1 - P \cdot 4(1 + 0.002P)^3 \cdot 0.002}{(1 + 0.002P)^8} \]

\[ \frac{dH}{dP} = \frac{81(1 + 0.002P)^3(1 + 0.002P - 0.008P)}{(1 + 0.002P)^8} \]

\[ \frac{dH}{dP} = \frac{81(1 - 0.006P)}{(1 + 0.002P)^5} \]

Solution (cont): The derivative is

\[ \frac{dH}{dP} = \frac{81(1 - 0.006P)}{(1 + 0.002P)^5} \]

- Critical points satisfy \( H'(P) = 0 \), so

\[ 1 - 0.006P = 0 \quad \text{or} \quad P = \frac{500}{3} = 166.7 \]

- With \( H(500/3) = 4271.5 \), the maximum occurs at

\[ (166.7, 4271.5) \]