

Calculus for the Life Sciences

Lecture Notes – Applications of the Derivative

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Outline

- 1 Application of The Derivative
 - Body Temperature Fluctuation
 - Critical Points
 - Maxima and Minima
 - Second Derivative and Concavity
 - Second Derivative Test
 - Points of Inflection
- 2 Examples
 - Cubic Polynomial
 - Quartic Polynomial
 - Absolute Maxima and Minima
 - Population Example



Applications of the Derivative

Applications of the Derivative

- Sketching Graphs
- Finding Maxima and Minima
- Optimal Values
- Points of Inflection or Steepest parts of a Function



Menstrual Cycle

Body Temperature Fluctuation
Critical Points
Maxima and Minima
Second Derivative and Concavity
Second Derivative Test
Points of Inflection

Body Temperature Fluctuation during the Menstrual Cycle

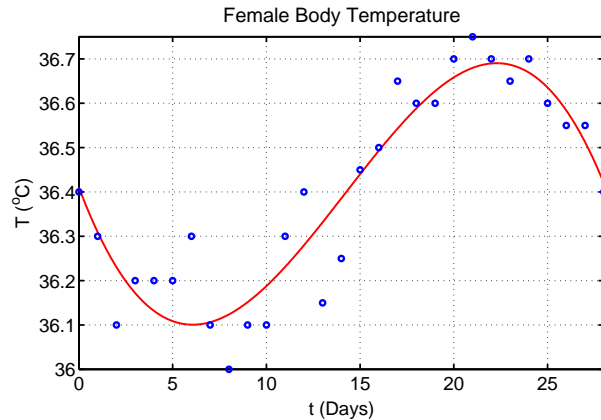
- Mammals regulate their body temperature in a narrow range to maintain optimal physiological responses
- Variations in body temperature occur during exercise, stress, infection, and other normal situations
- Neurological control of the temperature
- Variations include
 - Circadian rhythms (a few tenths of a degree Celsius)
 - Menstrual cycle - Ovulation often corresponds to the sharpest rise in temperature



Female Basal Body Temperature

1

Female Basal Body Temperature



Female Basal Body Temperature

2

Female Basal Body Temperature

- The best cubic polynomial fitting the data above is

$$T(t) = -0.0002762t^3 + 0.01175t^2 - 0.1121t + 36.41$$

- Want to find the high and low temperatures
- Determine the time of peak fertility when the temperature is rising most rapidly



Female Basal Body Temperature

3

Maximum and Minimum for

$$T(t) = -0.0002762t^3 + 0.01175t^2 - 0.1121t + 36.41$$

- The high and low temperatures occur when the curve has slope of zero
- Find the derivative equal to zero
- The derivative of the temperature is

$$T'(t) = -0.0008286t^2 + 0.02350t - 0.1121$$

- Note that the derivative is a different function from the original function



Female Basal Body Temperature

4

Maximum and Minimum

- The roots of the derivative (quadratic equation) are

$$t = 6.069 \quad \text{and} \quad 22.29 \text{ days}$$

- Thus,

Minimum at $t = 6.069$ with $T(6.069) = 36.1^\circ\text{C}$

Maximum at $t = 22.29$ with $T(22.29) = 36.7^\circ\text{C}$

- There is only a 0.6°C difference between the high and low basal body temperature during a 28 day menstrual cycle by the approximating function
- The data varied by 0.75°C



Female Basal Body Temperature

5

Maximum Increase in Temperature

- The maximum increase in temperature is when the derivative is at a maximum
- This is the vertex of the quadratic function, $T'(t)$
- The maximum occurs at the midpoint between the roots of the quadratic equation
- Alternately, the derivative of $T'(t)$ or the second derivative of $T(t)$ equal to zero gives the maximum
- The second derivative is

$$T''(t) = -0.0016572t + 0.02350$$



Female Basal Body Temperature

6

Maximum Increase in Temperature

- The second derivative is zero at the **Point of Inflection** at $t = 14.18$ with $T(14.18) = 36.4^\circ C$
- The maximum rate of change in body temperature is

$$T'(14.18) = 0.054^\circ C/day$$

- This model suggests that the peak fertility occurs on **day 14**, which is consistent with what is known about ovulation



Maxima, Minima, and Critical Points

1

Critical Points, Increasing and Decreasing

The derivative is zero at critical points for the graph of a **smooth function**

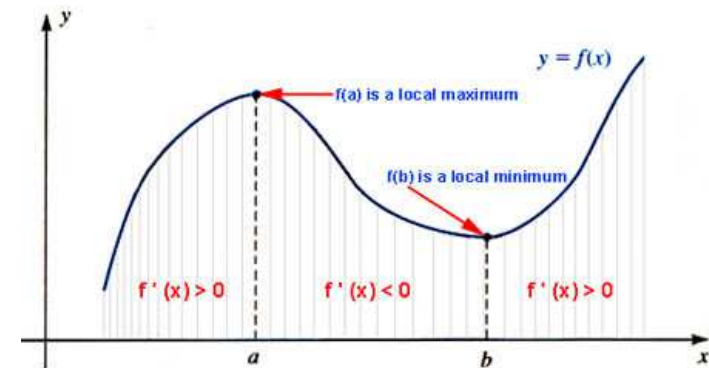
Definition: A smooth function $f(x)$ is said to be increasing on an interval (a, b) if $f'(x) > 0$ for all $x \in (a, b)$. Similarly, a smooth function $f(x)$ is said to be decreasing on an interval (a, b) if $f'(x) < 0$ for all $x \in (a, b)$



Maxima, Minima, and Critical Points

2

Maxima, Minima, and Critical Points



Maxima, Minima, and Critical Points

3

Maxima and Minima

- A high point of the graph is where $f(x)$ changes from increasing to decreasing
- A low point on a graph is where $f(x)$ changes from decreasing to increasing
- If $f(x)$ is a smooth function, then either case has the derivative passing through zero

Definition: A smooth function $f(x)$ is said to have a **local** or **relative maximum** at a point c , if $f'(c) = 0$ and $f'(x)$ changes from positive to negative for values of x near c . Similarly, a smooth function $f(x)$ is said to have a **local** or **relative minimum** at a point c , if $f'(c) = 0$ and $f'(x)$ changes from negative to positive for values of x near c .

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Maxima, Minima, and Critical Points

4

Critical Points

Definition: If $f(x)$ is a smooth function with $f'(x_c) = 0$, then x_c is said to be a **critical point** of $f(x)$

- Critical points help find the local high and low points on a graph
- Some critical points are neither maxima or minima

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Example – Graphing a Polynomial

1

Graphing a Polynomial Consider

$$f(x) = x^3 - 6x^2 - 15x + 2$$

- Find critical points, maxima, and minima
- Sketch a graph of the function

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Example – Graphing a Polynomial

2

With the function

$$f(x) = x^3 - 6x^2 - 15x + 2$$

we have the derivative

$$f'(x) = 3x^2 - 12x - 15 = 3(x + 1)(x - 5)$$

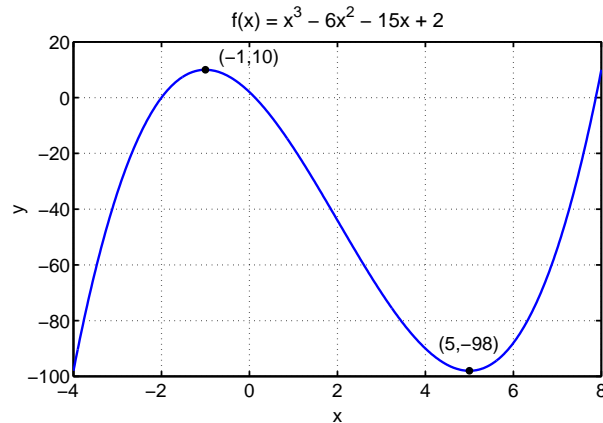
- The critical points ($f'(x) = 0$) are $x_c = -1$ or 5
- Since $f(-1) = 10$, a **local maximum** occurs at $(-1, 10)$
- Since $f(5) = -98$, a **local minimum** occurs at $(5, -98)$
- The y -intercept is $(0, 2)$

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Example – Graphing a Polynomial

3

This gives enough for a reasonable sketch of the graph
Note the x -intercepts are very hard to find



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Second Derivative and Concavity

Second Derivative and Concavity

- Since the derivative is itself a function, then if it is differentiable, one can take its derivative to find the **second derivative** often denoted $f''(x)$
- If the first derivative is increasing or the second derivative is positive, then the original function is **concave upward**
- If the first derivative is decreasing or the second derivative is negative, then the original function is **concave downward**
- The second derivative is a measure of the concavity of a function
- For smooth functions, the **maxima** generally occur where the function is **concave downward**, while **minima** occur where the function is **concave upward**

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Second Derivative Test

Second Derivative Test: Let $f(x)$ be a smooth function. Suppose that $f'(x_c) = 0$, so x_c is a critical point of f . If $f''(x_c) < 0$, then x_c is a **relative maximum**. If $f''(x_c) > 0$, then x_c is a **relative minimum**.

If $f''(x_c) = 0$, then we get **no information** about the function at the critical point x_c

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Example – Graphing a Polynomial

4

Continuing the example $f(x) = x^3 - 6x^2 - 15x + 2$

- The second derivative is $f''(x) = 6x - 12$
- Recall the critical points occurred at $x_c = -1$ and 5
- The second derivative at the critical point $x_c = -1$ gives

$$f''(-1) = -18$$

- The function is concave downward at -1 , so this is a relative maximum
- The second derivative at the critical point $x_c = 5$ gives

$$f''(5) = 18$$

- The function is concave upward, so this is a relative minimum

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Points of Inflection

Points of Inflection

- When the second derivative is zero, then the function is usually changing from concave upward to concave downward or visa versa
- This is known as a **point of inflection**
- A point of inflection is where the derivative function has a maximum or minimum, so the function is increasing or decreasing most rapidly
- From an applications point of view, if the function is describing a population, then the point of inflection would be where the population is increasing or decreasing most rapidly
- The point of inflection measures when the change of a function is its greatest or smallest



Example – Graphing a Polynomial

5

Continuing the example

$$f(x) = x^3 - 6x^2 - 15x + 2$$

- The second derivative is

$$f''(x) = 6x - 12$$

- The point of inflection is $f''(x) = 0$, which is when $x = 2$
- The point of inflection occurs at $(2, -44)$



Example – Graphing a Cubic Polynomial

1

Graphing a Polynomial: Consider

$$y(x) = 12x - x^3$$

- Find the intercepts
- Find critical points, maxima, and minima
- Find the points of inflection
- Sketch the graphs of the function, its derivative, and the second derivative

Skip Example



Example – Graphing a Cubic Polynomial

2

The **y-intercept** for $y(x) = 12x - x^3$ is **(0,0)**The **x-intercepts** satisfy

$$x(12 - x^2) = 0$$

$$x = 0, \pm 2\sqrt{3}$$



Example – Graphing a Cubic Polynomial

3

Maxima and Minima: The **derivative** of $y(x) = 12x - x^3$ is

$$y'(x) = 12 - 3x^2 = -3(x^2 - 4)$$

The **critical points** satisfy $y'(x_c) = 0$, so

$$x_c = -2 \quad \text{and} \quad x_c = 2$$

- Evaluating the original function at the critical points

$$y(-2) = -16 \quad \text{and} \quad y(2) = 16$$

- The extrema for the function are $(-2, -16)$ and $(2, 16)$
- Clearly, $(-2, -16)$ is a **minimum** and $(2, 16)$ is a **maximum**

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Example – Graphing a Cubic Polynomial

4

Point of Inflection: The **second derivative** of $y(x) = 12x - x^3$ is

$$y''(x) = -6x$$

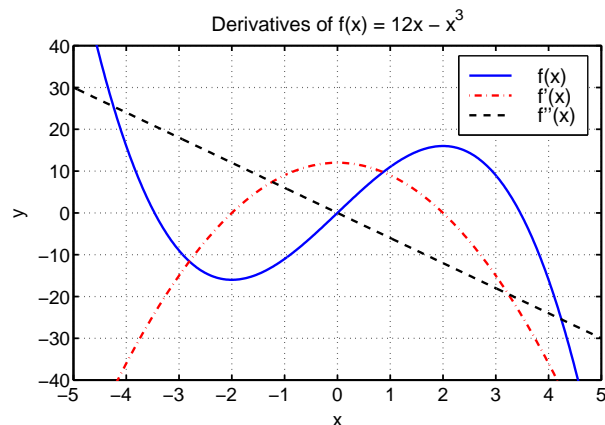
- The **second derivative test** gives
 - $y''(-2) = 12$ is concave upward, so $x_c = -2$ is a minimum
 - $y''(2) = -12$ is concave downward, so $x_c = 2$ is a maximum
- The **Point of Inflection** occurs at $x_p = 0$
- Concavity of the curve changes at $(0, 0)$

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Example – Graphing a Cubic Polynomial

5

Graphs of $y(x)$, $y'(x)$, and $y''(x)$



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Example – Graphing a Quartic Polynomial

1

Graphing a Polynomial: Consider

$$y(x) = x^4 - 8x^2$$

- Find the intercepts
- Find critical points, maxima, and minima
- Find the points of inflection
- Sketch the graphs of the function, its derivative, and the second derivative

Skip Example

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Example – Graphing a Quartic Polynomial

2

The **y-intercept** for $y(x) = x^4 - 8x^2$ is **(0,0)**

The **x-intercepts** satisfy

$$x^2(x^2 - 8) = 0$$

$$x = 0, \pm 2\sqrt{2}$$

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Example – Graphing a Quartic Polynomial

3

Maxima and Minima: The **derivative** of $y(x) = x^4 - 8x^2$ is

$$y'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$$

The **critical points** satisfy $y'(x_c) = 0$, so

$$x_c = -2, 0, 2$$

- Evaluating the original function at the critical points

$$y(-2) = -16, \quad y(0) = 0, \quad \text{and} \quad y(2) = -16$$

- The extrema for the function are $(-2, -16)$, $(0, 0)$, and $(2, -16)$
- Clearly, $(-2, -16)$ and $(2, -16)$ are **minima** and $(0, 0)$ is a **maximum**

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Example – Graphing a Quartic Polynomial

4

The **second derivative** of $y(x) = x^4 - 8x^2$ is

$$y''(x) = 12x^2 - 16$$

- The **second derivative test** gives
 - $y''(-2) = 32$ is concave upward, so $x_c = -2$ is a minimum
 - $y''(0) = -16$ is concave downward, so $x_c = 0$ is a maximum
 - $y''(2) = 32$ is concave upward, so $x_c = 2$ is a minimum

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Example – Graphing a Quartic Polynomial

5

Points of Inflection: Since the **second derivative** of $y(x) = x^4 - 8x^2$ is

$$y''(x) = 12x^2 - 16 = 4(3x^2 - 4)$$

- The **points of inflection** occur when $y''(x) = 0$
- $3x_p^2 - 4 = 0$, when

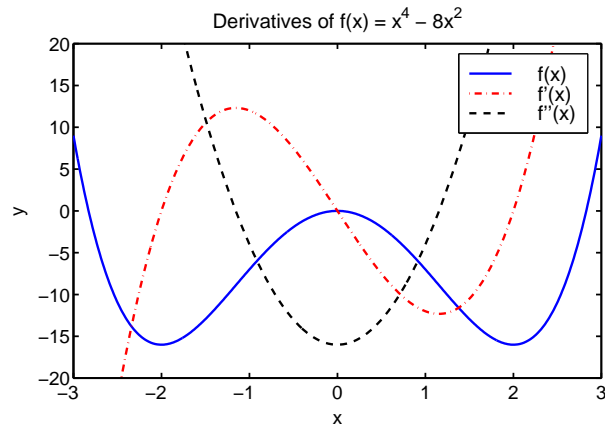
$$x_p = \pm \frac{2}{\sqrt{3}}$$

- Inflection points are $x_p \approx (\pm 1.155, -8.889)$

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Example – Graphing a Quartic Polynomial

6

Graphs of $y(x)$, $y'(x)$, and $y''(x)$ 

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Absolute Maxima and Minima

1

Absolute Maxima and Minima: Often we are interested in finding the largest or smallest population over a period of time

Definition: An **absolute minimum** for a function $f(x)$ occurs at a point $x = c$, if $f(c) \leq f(x)$ for all x in the domain of f .

Definition: Similarly, an **absolute maximum** for a function $f(x)$ occurs at a point $x = c$, if $f(c) \geq f(x)$ for all x in the domain of f .

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Absolute Maxima and Minima

2

Smooth functions on a closed interval always have an **absolute minimum** and an **absolute maximum**

Theorem: Suppose that $f(x)$ is a continuous, differentiable function on a closed interval $I = [a, b]$, then $f(x)$ achieves its **absolute minimum** (or **maximum**) on I and its minimum (or maximum) occurs either at a point where $f'(x) = 0$ or at one of the endpoints of the interval

This **theorem** says to find the function values at all the critical points and the endpoints of the interval, then this small set of values contains the **absolute minimum** and **absolute maximum**

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Example – Study of a Population

1

Study of a Population

- The ocean water is monitored for fecal contamination by counting certain types of bacteria in a sample of seawater
- Over a week where rain occurred early in the week, data were collected on one type of fecal bacteria
- The population of the particular bacteria (in thousand/cc), $P(t)$, were best fit by the cubic polynomial

$$P(t) = -t^3 + 9t^2 - 15t + 40,$$

where t is in days

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Example – Study of a Population

2

Study of a Population

- Find the rate of change in population per day, dP/dt
- What is the rate of change in the population on the third day?
- Find relative and absolute minimum and maximum populations of the bacteria over the time of the surveys
- Determine when the bacterial count is most rapidly increasing
- Sketch a graph of this polynomial fit to the population of bacteria
- When did the rain most likely occur?

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Example – Study of a Population

3

Rate of Change in Population: The derivative of $P(t) = -t^3 + 9t^2 - 15t + 40$ is

$$\frac{dP}{dt} = -3t^2 + 18t - 15$$

Evaluating this on the third day

$$\frac{dP(3)}{dt} = 12(\times 1000/\text{cc}/\text{day})$$

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Example – Study of a Population

4

Critical Points: We found

$$\frac{dP}{dt} = -3t^2 + 18t - 15 = -3(t-1)(t-5)$$

- The **critical points** are
 - $t_c = 1$ and $t_c = 5$
 - **Relative Minimum** at $t_c = 1$ with $P(1) = 33(\times 1000/\text{cc})$
 - **Relative Maximum** at $t_c = 5$ with $P(5) = 65(\times 1000/\text{cc})$
- The endpoint values are $P(0) = 40$ and $P(7) = 33$
- By the theorem above
 - The **absolute maximum** occurs at $t = 5$ with $P(5) = 65$
 - The **absolute minimum** occurs at $t = 1$ and 7 with $P(1) = P(7) = 33$

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Example – Study of a Population

5

Point of Inflection: The bacteria is increasing most rapidly when the second derivative is zero

Since $P'(t) = -3t^2 + 18t - 15$, the second derivative is

$$P''(t) = -6t + 18 = -6(t-3)$$

- The population is increasing most rapidly at $t = 3$ with $P(3) = 49(\times 1000/\text{cc})$
- This maximum increase is

$$P'(3) = 12(\times 1000/\text{cc}/\text{day})$$

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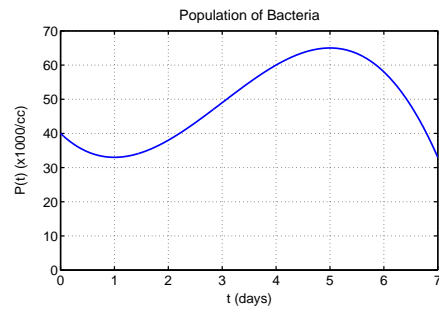
Example – Study of a Population

6

Graph of

$$P(t) = -t^3 + 9t^2 - 15t + 40,$$

From the graph, we can guess that the rain fell on the second day of the week with storm runoff polluting the water in the days following



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