In the Introduction to the lecture notes, we referred to the data from the Hudson Bay Company. This graph suggests that the population of snowshoe hares follows a periodic function, although it is not a true sine or cosine wave, due to their interaction with their primary predator the lynx. We noted in the beginning of the lecture section on differentiation of trigonometric functions that the work of Fourier showed that any continuous, periodic function can be modeled with the infinite summation of a series of sine and/or cosine waves.

In this question you will use the work of Fourier and Excel’s solver, to model the quantity of the hare pelts (in thousands) that were turned into the Hudson Bay Trading Company for years after 1900. The data for hare pelts (in thousands) is shown below:

<table>
<thead>
<tr>
<th>Year</th>
<th>Hare Pelts</th>
</tr>
</thead>
<tbody>
<tr>
<td>1900</td>
<td>30</td>
</tr>
<tr>
<td>1901</td>
<td>47.2</td>
</tr>
<tr>
<td>1902</td>
<td>70.2</td>
</tr>
<tr>
<td>1903</td>
<td>77.4</td>
</tr>
<tr>
<td>1904</td>
<td>36.3</td>
</tr>
<tr>
<td>1905</td>
<td>20.6</td>
</tr>
<tr>
<td>1906</td>
<td>18.1</td>
</tr>
</tbody>
</table>

We need to find the appropriate parameters amplitude, $a_n$, frequency, $\omega$, and principle phase shift, $\phi_n$, which best fit the data. Choose the unique parameters amplitude, $a_n > 0$, frequency, $\omega > 0$, and principle phase shift, $\phi_n \in [0, T]$, where $T$ is the period of the appropriate trigonometric function. Note that Excel’s Solver may select $a_n < 0$ or $\phi_n$ outside the interval $[0, T]$. Use the periodic properties of the trigonometric functions to obtain the unique parameters described above.

**Part a.** Begin by trying to fit the data with a simple sine function and a constant, which satisfies the equation

$$P(t) = a_0 + a_1 \sin(\omega(t - \phi_1)).$$

where $t = 0$ corresponds to 1900. Use Excel’s solver to find the constants $a_0$, $a_1$, $\omega$, and $\phi_1$, which give a least squares best fit to the data. For initial guesses, take $a_0$ to be the average of the data and $a_1$ be the difference between the maximum of the data and the average. Use the difference in years between the two maxima in the data to approximate the period and use that to estimate $\omega$. Let $\phi_1 = 0$.

$$a_0 = \ldots$$
$$a_1 = \ldots$$
$$\omega = \ldots$$
$$\phi_1 = \ldots$$

Find the percent error between the model and the data for $t = 4$ (1904), $t = 9$ (1909), $t = 13$ (1913), and $t = 18$ (1918).

$$P(4) = \ldots$$
$$P(9) = \ldots$$
$$P(13) = \ldots$$
$$P(18) = \ldots$$

Find the values of $t$, where the absolute minimum and absolute maximum occur, and values $P(t)$ of the absolute minimum and the absolute maximum from this approximation. (Give only the first occurrence as it is periodic.)

Minimum at $t = \ldots$ with $P(t_{\text{min}}) = \ldots$
Maximum at $t = \ldots$ with $P(t_{\text{max}}) = \ldots$

**Part b.** The next step in this problem is to see how much better the data are fit using another sine function in the Fourier series. The second sine function has double the frequency of the first (sharing $\omega$). Thus, we want to fit the function

$$P(t) = a_0 + a_1 \sin(\omega(t - \phi_1)) + a_2 \sin(2\omega(t - \phi_2)).$$

Use Excel’s solver to find the constants $a_0$, $a_1$, $a_2$, $\omega$, $\phi_1$, and $\phi_2$, which give a least squares best fit to the data. For initial guesses, use your best fitting values of $a_0$, $a_1$, $\omega$, and $\phi_1$ from Part a. Take initial guesses of $a_2 = 1$ and $\phi_2 = 0$. 
Find the percent error between the model and the data for $t = 4$ (1904), $t = 9$ (1909), $t = 13$ (1913), and $t = 18$ (1918).

$P(4) = ____ \text{ with Percent Error = _____}$

$P(9) = ____ \text{ with Percent Error = _____}$

$P(13) = ____ \text{ with Percent Error = _____}$

$P(18) = ____ \text{ with Percent Error = _____}$

d. Repeat the process in Part c. adding a fourth term term, $a_4 \sin(4\omega(t - \phi_4))$.

Use Excel’s solver to find the constants $a_0$, $a_1$, $a_2$, $a_3$, $a_4$, $\omega$, $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$, which give a least squares best fit to the data. For initial guesses, use your best fitting values of $a_0$, $a_1$, $a_2$, $a_3$, $\omega$, $\phi_1$, $\phi_2$ and $\phi_3$ from Part c. Take initial guesses of $a_4 = 1$ and $\phi_4 = 0$.

Use Excel’s solver to find the constants $a_0$, $a_1$, $a_2$, $a_3$, $\omega$, $\phi_1$, $\phi_2$ and $\phi_3$, which give a least squares best fit to the data. For initial guesses, use your best fitting values of $a_0$, $a_1$, $a_2$, $\omega$, $\phi_1$ and $\phi_2$ from Part b. Take initial guesses of $a_3 = 1$ and $\phi_3 = 0$.
\[ P(4) = \quad \text{with Percent Error} = \quad \]
\[ P(9) = \quad \text{with Percent Error} = \quad \]
\[ P(13) = \quad \text{with Percent Error} = \quad \]
\[ P(18) = \quad \text{with Percent Error} = \quad \]

Find the values of \( t \), where the absolute minimum and absolute maximum occur, and values \( P(t) \) of the absolute minimum and the absolute maximum from this approximation. (Give only the first occurrence as it is periodic.)

Minimum at \( t = \quad \) with \( P(t_{\text{min}}) = \quad \)
Maximum at \( t = \quad \) with \( P(t_{\text{max}}) = \quad \)

\[ e. \quad \text{In your Lab Report, on a single graph plot the data and all four models from } t \in [0, 20], \text{ (1900 to 1920). Describe in some detail how each of the four models fits the data. Discuss the evolution from the first model to the fourth model on how the models compare to the data and what changes are observed in the parameters. Describe relative changes in the shared parameters for the sequence of models, such as the evolution of the amplitude, frequency, or phase shift. Do these models have the same period? What is the relative size (think amplitude) of the new terms in the subsequent models to the earlier models? That is discuss how much the different coefficients change as you add more terms to the series. Describe the shapes of the combined models and how well they fit the data. Write a brief discussion of what is the trend for the sum of least squares error and for the calculated percent error at the dates requested? Do you expect a much better fit by adding a fifth term to the series? Write a short paragraph describing adjustments that you needed to obtain the correct parameters with Excel’s Solver. Describe how you obtained the initial estimates, then explain how you made adjustments to obtain positive amplitudes and phase shifts in the appropriate range. In addition, describe the methods you used to find the absolute maxima and minima. Give details of the mathematical and numerical methods needed to find these critical points.}

As described in the paragraph beginning this problem, these data come from a collection of pelts for snowshoe hares and lynx from the Hudson Bay trading company. These animals occupy a classical case of a tightly woven pair of prey animals and their primary predators. Write a paragraph on why these animal populations might exhibit an oscillatory relationship. Give some details on what you might expect on the relative temporal population changes of these populations. Specifically, take your data set for hare above and describe what you would expect for the population of the other animal, the lynx. Specifically, can you speculate the comparative relationships (in time) of the maxima and minima? Would the shape or period vary much between the population studies of these animals? Give a brief explanation.