Math 636 – Mathematical Modeling

Lecture Notes – More Applications of Nonlinear Dynamical Systems

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Outline

1. Introduction
2. Salmon Populations
   - Ricker's Model
3. Analysis of the Ricker's Model
   - Equilibria
   - Stability Analysis
   - Skeena River Salmon Example
4. Beverton-Holt and Hassell's Model
   - Study of a Beetle Population
   - Analysis of Hassell's Model
   - Beetle Study Analysis

Introduction - Population Models

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- Differentiation needed to analyze these models

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- Their bodies provide many essential nutrients that nourish the stream of their young

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- Simplify the model by ignoring this complexity

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Sockeye Salmon Populations – Skeena River Table

Population in thousands

Year	Population	Year	Population
1908	1,098	1932	278
1912	740	1936	448
1916	714	1940	528
1920	615	1944	639
1924	706	1948	523
1928	510		

Four Year Averages of Skeena River Sockeye Salmon

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- Fishery management has often used **Ricker's Model**

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- Consider the Skeena river salmon data
  - The parent population of 1908-1911 is averaged to 1,098,000 salmon/year returning to the Skeena river watershed
  - It is assumed that the resultant offspring that return to spawn from this group occurs between 1912 and 1915, which averages 740,000 salmon/year

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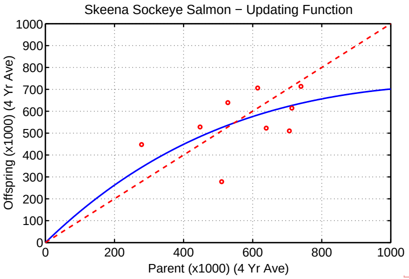
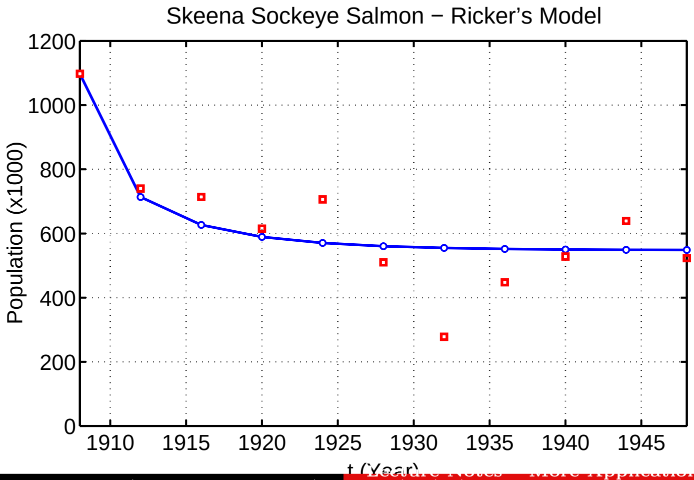
- Successive populations give data for updating functions
  - P_n is parent population, and P_{n+1} is surviving offspring
  - Nonlinear least squares fit of Ricker's function

$$P_{n+1} = 1.535\, P_n e^{-0.000783\, P_n}$$



Ricker's Model – Salmon

Simulate the Ricker's model using the initial average in 1908 as a starting point



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- There are a few fluctuations from the variations in the environment
- Low point during depression, suggesting bias from economic factors

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Analysis of the Ricker's Model: General Ricker's Model

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Note that $a > 1$ required for a positive equilibrium

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Stability Analysis of the Ricker's Model: Find the derivative of the updating function

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- If $a > 1$, then $P_e = 0$ is unstable and the population grows away from the equilibrium

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- The solution of Ricker's model is **unstable** and **oscillates as it grows away** the equilibrium $P_e = \ln(a)/b$ provided $a > e^2 \approx 7.389$

Skeena River Salmon Example

The best Ricker's model for the Skeena sockeye salmon population from 1908-1952 is

$$P_{n+1} = R(P_n) = 1.535\, P_n e^{-0.000783\, P_n}$$

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- At $P_e = 547.3$, $R'(547.3) = 0.571 < 1$
  - This equilibrium is **stable** with solutions monotonically approaching the equilibrium, as observed in the simulation

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Beverton-Holt Model - Rational form

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- Rare amongst nonlinear models - Has an explicit solution
- Given an initial population, P_0

$$P_{n+1} = \frac{MP_0}{P_0 + (M - P_0)a^{-n}}$$

Hassell's Model

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- Often used in insect populations
- Provides alternative to **logistic** and **Ricker's** growth models, extending the **Beverton-Holt** model
- $H(P_n)$ has **3 parameters**, a, b, and c, while logistic, Ricker's, and Beverton-Holt models have **2 parameters**

Hassell's Model

Hassell's Model - Alternate Rational form

$$P_{n+1} = H(P_n) = \frac{aP_n}{(1 + bP_n)^c}$$

- Often used in insect populations
- Provides alternative to **logistic** and **Ricker's** growth models, extending the **Beverton-Holt** model
- $H(P_n)$ has **3 parameters**, a, b, and c, while logistic, Ricker's, and Beverton-Holt models have **2 parameters**
- Malthusian growth rate $a - 1$, like Beverton-Holt model

Study of a Beetle Population 1

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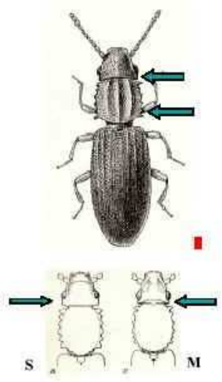
Study of a Beetle Population

Study of a Beetle Population

- In 1946, A. C. Crombie studied several beetle populations
- The food was strictly controlled to maintain a constant supply
- 10 grams of cracked wheat were added weekly
- Regular census of the beetle populations recorded
- These are experimental conditions for the **Logistic growth model**

Study of a Beetle Population

Study of *Oryzaephilus surinamensis*, **the saw-tooth grain beetle**



Gorham, 1967

Study of a Beetle Population

Data on *Oryzaephilus surinamensis*, **the saw-tooth grain beetle**

Week	Adults	Week	Adults
0	4	16	405
2	4	18	471
4	25	20	420
6	63	22	430
8	147	24	420
10	285	26	475
12	345	28	435
14	361	30	480

Study of a Beetle Population 4

Updating functions - Least squares best fit to data

- Plot the data, P_{n+1} vs. P_n, to fit an updating function

Study of a Beetle Population

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$$P_{n+1} = P_n + 0.962\, P_n \left(1 - \frac{P_n}{439.2}\right)$$

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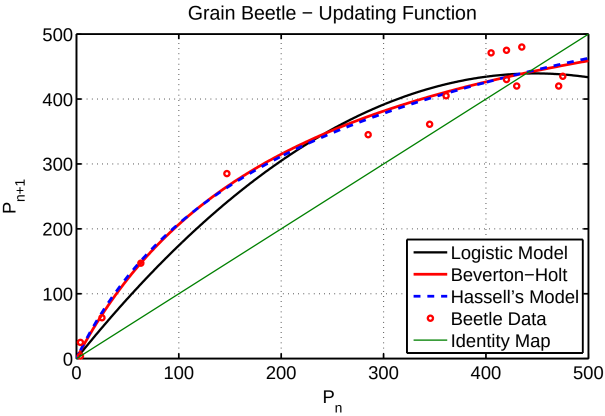
- **Hassell's growth model** fit to data (SSE = 9,955)

$$P_{n+1} = \frac{3.269\, P_n}{(1 + 0.00745\, P_n)^{0.8126}}$$

Study of a Beetle Population

Graph of **Updating functions** and **Beetle data**



Study of a Beetle Population 6

Time Series - Least squares best fit to data, P_0

- Use the **updating functions** from fitting data before
- Adjust P_0 by **least sum of square errors** to time series data on beetles

Study of a Beetle Population

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Study of a Beetle Population

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Study of a Beetle Population

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Study of a Beetle Population

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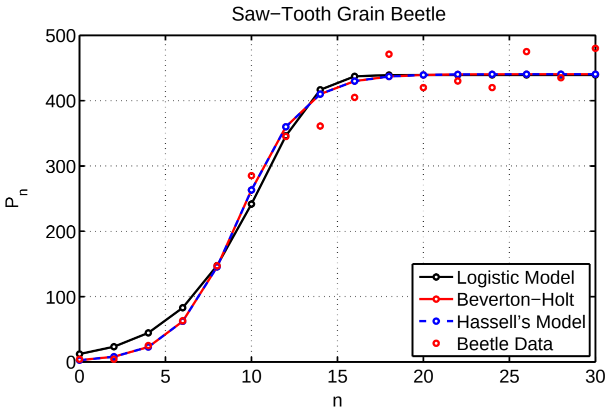
- **Hassell's growth model** fit to data gives $P_0 = 2.08$ with SSE $= 7{,}948$
- Beverton-Holt and Hassell's models are very close with both significantly better than the logistic growth model

Study of a Beetle Population

Time Series graph of Models with Beetle Data



Analysis of Hassell's Model 1

Analysis of Hassell's Model – Equilibria

- Let $P_e = P_{n+1} = P_n$, so

$$P_e = \frac{aP_e}{(1 + bP_e)^c}$$

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Analysis of Hassell's Model – Equilibria

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- One equilibrium is $P_e = 0$ (as expected the extinction equilibrium)
- The other satisfies

$$\begin{aligned} (1+bP_e)^c &= a \\ 1+bP_e &= a^{1/c} \\ P_e &= \frac{a^{1/c}-1}{b} \end{aligned}$$

Analysis of Hassell's Model 2

Analysis of Hassell's Model – Stability Analysis

- Hassell's updating function is

$$H(P) = \frac{aP}{(1+bP)^c}$$

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- The derivative of the denominator (chain rule) is

$$\frac{d}{dP}(1+bP)^c = c(1+bP)^{c-1}b = bc(1+bP)^{c-1}$$

Analysis of Hassell's Model 2

Analysis of Hassell's Model – Stability Analysis

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$$\frac{d}{dP}(1+bP)^c = c(1+bP)^{c-1}b = bc(1+bP)^{c-1}$$

- By the quotient rule

$$\begin{aligned} H'(P) &= \frac{a(1+bP)^c - abcP(1+bP)^{c-1}}{(1+bP)^{2c}} \\ &= a\frac{1+b(1-c)P}{(1+bP)^{c+1}} \end{aligned}$$

Analysis of Hassell's Model

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Analysis of Hassell's Model 3

Analysis of Hassell's Model – Stability Analysis

- The derivative is

$$H'(P) = a\frac{1 + b(1-c)P}{(1+bP)^{c+1}}$$

- At $P_e = 0$, $H'(0) = a$
  - Since $a > 1$, the zero equilibrium is **unstable**
  - Solutions **monotonically growing away** from the **extinction equilibrium**

Analysis of Hassell's Model 4

Analysis of Hassell's Model – Stability Analysis

- The derivative is

$$H'(P) = a\frac{1 + b(1-c)P}{(1+bP)^{c+1}}$$

Analysis of Hassell's Model 4

Analysis of Hassell's Model – Stability Analysis

- The derivative is

$$H'(P) = a\frac{1+b(1-c)P}{(1+bP)^{c+1}}$$

- At $P_e = (a^{1/c}-1)/b$, we find

$$\begin{aligned} H'(P_e) &= a\frac{1+(1-c)(a^{1/c}-1)}{(1+(a^{1/c}-1))^{c+1}} \\ &= \frac{c}{a^{1/c}} + 1 - c \end{aligned}$$

Analysis of Hassell's Model

Analysis of Hassell's Model – Stability Analysis

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  - The stability of the **carrying capacity equilibrium** depends on both a and c, but not b

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  - The stability of the **carrying capacity equilibrium** depends on both a and c, but not b
  - When $c = 1$ (**Beverton-Holt** model) $H'(P_e) = \frac{1}{a}$, so this equilibrium is **monotonically stable**

Beetle Study Analysis 1

Beetle Study Analysis – Logistic Growth Model

$$P_{n+1} = F(P_n) = P_n + 0.962\, P_n \left(1 - \frac{P_n}{439.2}\right)$$

- The **equilibria** are $P_e = 0$ and 439.2

Beetle Study Analysis

Beetle Study Analysis – Logistic Growth Model

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- The **equilibria** are $P_e = 0$ and 439.2
- The derivative of the updating function is

$$F'(P) = 1.962 - 0.00438\, P$$

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- The **equilibria** are $P_e = 0$ and 439.2
- The derivative of the updating function is

$$F'(P) = 1.962 - 0.00438\, P$$

  - At $P_e = 0$, $F'(0) = 1.962$, so this equilibrium is **monotonically unstable**

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$$P_{n+1} = F(P_n) = P_n + 0.962\, P_n \left(1 - \frac{P_n}{439.2}\right)$$

- The **equilibria** are $P_e = 0$ and 439.2
- The derivative of the updating function is

$$F'(P) = 1.962 - 0.00438\, P$$

- At $P_e = 0$, $F'(0) = 1.962$, so this equilibrium is **monotonically unstable**
- At $P_e = 439.2$, $F'(439.2) = 0.038$, so this equilibrium is **monotonically stable**

Beetle Study Analysis 2

Beetle Study Analysis – Beverton-Holt Growth Model

$$P_{n+1} = B(P_n) = \frac{3.010\, P_n}{1 + 0.00456\, P_n}$$

- The **equilibria** are $P_e = 0$ and 440.8

Beetle Study Analysis 2

Beetle Study Analysis – Beverton-Holt Growth Model

$$P_{n+1} = B(P_n) = \frac{3.010\, P_n}{1 + 0.00456\, P_n}$$

- The **equilibria** are $P_e = 0$ and 440.8
- The derivative of the updating function is

$$B'(P) = \frac{3.010}{(1 + 0.00456\, P)^2}$$

Beetle Study Analysis 2

Beetle Study Analysis – Beverton-Holt Growth Model

$$P_{n+1} = B(P_n) = \frac{3.010\, P_n}{1 + 0.00456\, P_n}$$

- The **equilibria** are $P_e = 0$ and 440.8
- The derivative of the updating function is

$$B'(P) = \frac{3.010}{(1 + 0.00456\, P)^2}$$

  - At $P_e = 0$, $B'(0) = 3.010$, so this equilibrium is **monotonically unstable**

Beetle Study Analysis

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- The **equilibria** are $P_e = 0$ and 440.8
- The derivative of the updating function is

$$B'(P) = \frac{3.010}{(1 + 0.00456\,P)^2}$$

  - At $P_e = 0$, $B'(0) = 3.010$, so this equilibrium is **monotonically unstable**
  - At $P_e = 440.8$, $B'(440.8) = 0.332$, so this equilibrium is **monotonically stable**

Beetle Study Analysis

Beetle Study Analysis – Hassell's Growth Model

$$P_{n+1} = H(P_n) = \frac{3.269\, P_n}{(1 + 0.00745\, P_n)^{0.8126}}$$

- The **equilibria** are $P_e = 0$ and 442.4

Beetle Study Analysis

Beetle Study Analysis – Hassell's Growth Model

$$P_{n+1} = H(P_n) = \frac{3.269\,P_n}{(1+0.00745\,P_n)^{0.8126}}$$

- The **equilibria** are $P_e = 0$ and 442.4
- The derivative of the updating function is

$$H'(P) = 3.269\frac{1+0.001396\,P}{(1+0.00745P)^{1.8126}}$$

Beetle Study Analysis

Beetle Study Analysis – Hassell's Growth Model

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- The derivative of the updating function is

$$H'(P) = 3.269 \frac{1 + 0.001396\, P}{(1 + 0.00745P)^{1.8126}}$$

  - At $P_e = 0$, $H'(0) = 3.269$, so this equilibrium is **monotonically unstable**

Beetle Study Analysis

Beetle Study Analysis – Hassell's Growth Model

$$P_{n+1} = H(P_n) = \frac{3.269\, P_n}{(1 + 0.00745\, P_n)^{0.8126}}$$

- The **equilibria** are $P_e = 0$ and 442.4
- The derivative of the updating function is

$$H'(P) = 3.269 \frac{1 + 0.001396\, P}{(1 + 0.00745 P)^{1.8126}}$$

  - At $P_e = 0$, $H'(0) = 3.269$, so this equilibrium is **monotonically unstable**
  - At $P_e = 442.4$, $H'(442.4) = 0.3766$, so this equilibrium is **monotonically stable**