Calculus for the Life Sciences II
Lecture Notes – Linear Differential Equations

Joseph M. Mahaffy,
(mahaffy@math.sdsu.edu)

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://www-rohan.sdsu.edu/~jmahaffy

Fall 2012
Outline

1. Introduction
2. Blood Pressure
   - Cardiac Cycle
   - Arterial Blood Pressure
   - Modeling Blood Pressure
   - Diagnosis with Model
   - Example of Athlete
3. Radioactive Decay
   - Carbon Radiodating
   - Hyperthyroidism
4. Solution of Linear Growth and Decay Models
5. Newton’s Law of Cooling
   - Murder Investigation
   - Cooling Tea
6. Solution of General Linear Model
7. Pollution in a Lake
   - Example of Pollution with Evaporation
Introduction

- Examples of linear first order differential equations
Introduction

- Examples of linear first order differential equations
  - Arterial blood pressure
Introduction

Examples of linear first order differential equations
- Arterial blood pressure
- Radioactive decay
Introduction

- Examples of linear first order differential equations
  - Arterial blood pressure
  - Radioactive decay
  - Newton’s law of cooling
Introduction

Examples of linear first order differential equations

- Arterial blood pressure
- Radioactive decay
- Newton’s law of cooling
- Pollution in a Lake
Introduction

Examples of linear first order differential equations
- Arterial blood pressure
- Radioactive decay
- Newton’s law of cooling
- Pollution in a Lake

Extend earlier techniques to find solutions
Blood Pressure

- **Blood Pressure** is divided into **systolic** and **diastolic** pressure
Blood Pressure

- Blood Pressure is divided into systolic and diastolic pressure
- Normal reading is 120/80 (in mm of Hg)
Blood Pressure

- **Blood Pressure** is divided into **systolic** and **diastolic** pressure.
- Normal reading is **120/80** (in mm of Hg).
- How are those numbers generated and what can we infer from them?
Blood Pressure

- **Blood Pressure** is divided into **systolic** and **diastolic** pressure.
- Normal reading is **120/80** (in mm of Hg).
- How are those numbers generated and what can we infer from them?
- The numbers for blood pressure reflect the force on arterial walls.
Blood Pressure

- **Blood Pressure** is divided into **systolic** and **diastolic** pressure.
- Normal reading is **120/80** (in mm of Hg).
- How are those numbers generated and what can we infer from them?
- The numbers for blood pressure reflect the force on arterial walls.
- This pressure is generated by the beating of the heart.
Blood Pressure

Diagram of Heart
Cardiac Cycle

- Pulmonary circulation
Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
  - Flows to the right ventricle
Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
  - Flows to the right ventricle
  - Blood goes through the pulmonary artery to the lungs
Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
  - Flows to the right ventricle
  - Blood goes through the pulmonary artery to the lungs
  - Blood exchanges $\text{O}_2$ and $\text{CO}_2$ in the lungs
Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
  - Flows to the right ventricle
  - Blood goes through the pulmonary artery to the lungs
  - Blood exchanges $O_2$ and $CO_2$ in the lungs
  - Blood returns through the pulmonary vein to the left atrium
Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
  - Flows to the right ventricle
  - Blood goes through the pulmonary artery to the lungs
  - Blood exchanges O\textsubscript{2} and CO\textsubscript{2} in the lungs
  - Blood returns through the pulmonary vein to the left atrium
  - Pressure in the pulmonary vein and left atrium is between 5 and 15 mm of Hg
Cardiac Cycle

Pulmonary circulation

- Blood flows from the body into the right atrium
- Flows to the right ventricle
- Blood goes through the pulmonary artery to the lungs
- Blood exchanges $O_2$ and $CO_2$ in the lungs
- Blood returns through the pulmonary vein to the left atrium
- Pressure in the pulmonary vein and left atrium is between 5 and 15 mm of Hg

- Blood flows into the left ventricle
Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
  - Flows to the right ventricle
  - Blood goes through the pulmonary artery to the lungs
  - Blood exchanges $O_2$ and $CO_2$ in the lungs
  - Blood returns through the pulmonary vein to the left atrium
  - Pressure in the pulmonary vein and left atrium is between 5 and 15 mm of Hg
- Blood flows into the left ventricle
- The heart is rigid, so pressure increases only slightly
Cardiac Cycle

- Pulmonary circulation
  - Blood flows from the body into the right atrium
  - Flows to the right ventricle
  - Blood goes through the pulmonary artery to the lungs
  - Blood exchanges $O_2$ and $CO_2$ in the lungs
  - Blood returns through the pulmonary vein to the left atrium
  - Pressure in the pulmonary vein and left atrium is between 5 and 15 mm of Hg
- Blood flows into the left ventricle
- The heart is rigid, so pressure increases only slightly
- The right atrium contracts, then the AV valve between the atrium and the ventricle closes
Cardiac Cycle (cont)

- The heart receives an electrical signal, which causes ventricular contraction, beginning **systole**
Cardiac Cycle (cont)

- The heart receives an electrical signal, which causes ventricular contraction, beginning **systole**

- The left ventricle contracts, and the pressure increases until it “blows” open the aortic valve
Cardiac Cycle (cont)

- The heart receives an electrical signal, which causes ventricular contraction, beginning *systole*

- The left ventricle contracts, and the pressure increases until it “blows” open the aortic valve

- Blood rapidly flows into the aorta under this high pressure (*systolic pressure*)
Cardiac Cycle (cont)

- The heart receives an electrical signal, which causes ventricular contraction, beginning systole.

- The left ventricle contracts, and the pressure increases until it “blows” open the aortic valve.

- Blood rapidly flows into the aorta under this high pressure (systolic pressure).

- As pressure rises in the aorta, the AV valve reopens, while the aortic valve closes.
Cardiac Cycle (cont)

- The heart receives an electrical signal, which causes ventricular contraction, beginning **systole**
- The left ventricle contracts, and the pressure increases until it “blows” open the aortic valve
- Blood rapidly flows into the aorta under this high pressure (**systolic pressure**)
- As pressure rises in the aorta, the AV valve reopens, while the aortic valve closes
- Now there is high pressure in the aorta, which forces the blood into the other arteries of the body
Cardiac Cycle (cont)

- The heart receives an electrical signal, which causes ventricular contraction, beginning **systole**
- The left ventricle contracts, and the pressure increases until it “blows” open the aortic valve
- Blood rapidly flows into the aorta under this high pressure (**systolic pressure**)
- As pressure rises in the aorta, the AV valve reopens, while the aortic valve closes
- Now there is high pressure in the aorta, which forces the blood into the other arteries of the body
- As the blood flows through the body, the aortic pressure drops to its low pressure, the **diastolic pressure**
Cardiac Cycle (cont)

- The heart receives an electrical signal, which causes ventricular contraction, beginning **systole**
- The left ventricle contracts, and the pressure increases until it “blows” open the aortic valve
- Blood rapidly flows into the aorta under this high pressure (**systolic pressure**)
- As pressure rises in the aorta, the AV valve reopens, while the aortic valve closes
- Now there is high pressure in the aorta, which forces the blood into the other arteries of the body
- As the blood flows through the body, the aortic pressure drops to its low pressure, the **diastolic pressure**
Arterial Blood Pressure

**Arterial Blood Pressure:** Model the arterial pressure, $P_a(t)$, during a single beat of the heart
**Arterial Blood Pressure:** Model the arterial pressure, $P_a(t)$, during a single beat of the heart

- Determine the important modeling parameters in the system
Arterial Blood Pressure: Model the arterial pressure, $P_a(t)$, during a single beat of the heart

- Determine the important modeling parameters in the system
- The cardiac output, $Q$, represents the average amount of blood pumped by the heart (in liters/min)
Arterial Blood Pressure: Model the arterial pressure, $P_a(t)$, during a single beat of the heart

- Determine the important modeling parameters in the system
- The cardiac output, $Q$, represents the average amount of blood pumped by the heart (in liters/min)
- The stroke volume, $V$, is the amount of blood pumped by the heart during one beat (liters/beat)
Arterial Blood Pressure: Model the arterial pressure, $P_a(t)$, during a single beat of the heart

- Determine the important modeling parameters in the system
- The cardiac output, $Q$, represents the average amount of blood pumped by the heart (in liters/min)
- The stroke volume, $V$, is the amount of blood pumped by the heart during one beat (liters/beat)
- $T$ is the duration of a heart beat
Arterial Blood Pressure: Model the arterial pressure, $P_a(t)$, during a single beat of the heart

- Determine the important modeling parameters in the system
- The cardiac output, $Q$, represents the average amount of blood pumped by the heart (in liters/min)
- The stroke volume, $V$, is the amount of blood pumped by the heart during one beat (liters/beat)
- $T$ is the duration of a heart beat
- Relate flow from the cardiac output to the stroke volume by the relationship

$$\text{Cardiac Output} = \frac{\text{Stroke Volume}}{\text{Duration of the Flow}}$$

$$Q = \frac{V}{T} \text{ (liters/min)}$$
Arterial Blood Pressure: Model the arterial pressure, $P_a(t)$, during a single beat of the heart

- Determine the important modeling parameters in the system
- The cardiac output, $Q$, represents the average amount of blood pumped by the heart (in liters/min)
- The stroke volume, $V$, is the amount of blood pumped by the heart during one beat (liters/beat)
- $T$ is the duration of a heart beat
- Relate flow from the cardiac output to the stroke volume by the relationship

Cardiac Output = Stroke Volume / Duration of the Flow

$$Q = V/T \text{ (liters/min)}$$
Arterial Blood Pressure:
Arterial Blood Pressure:

- The left ventricle completes pumping the blood into the aorta and the aortic valve closes at the maximum pressure, $P_{sys}$
Arterial Blood Pressure:

- The left ventricle completes pumping the blood into the aorta and the aortic valve closes at the maximum pressure, $P_{\text{sys}}$.
- The blood pressure begins to fall as the blood flows through the arteries.
Arterial Blood Pressure:

- The left ventricle completes pumping the blood into the aorta and the aortic valve closes at the maximum pressure, $P_{sys}$.
- The blood pressure begins to fall as the blood flows through the arteries.
- The rate of flowing of the blood depends on the resistance of a blood vessel.
Arterial Blood Pressure:

- The left ventricle completes pumping the blood into the aorta and the aortic valve closes at the maximum pressure, $P_{sys}$
- The blood pressure begins to fall as the blood flows through the arteries
- The rate of flowing of the blood depends on the resistance of a blood vessel
  - Viscosity of the blood
Arterial Blood Pressure:

- The left ventricle completes pumping the blood into the aorta and the aortic valve closes at the maximum pressure, $P_{sys}$.
- The blood pressure begins to fall as the blood flows through the arteries.
- The rate of flowing of the blood depends on the **resistance** of a blood vessel.
  - Viscosity of the blood
  - Length of the vessels
Arterial Blood Pressure:

- The left ventricle completes pumping the blood into the aorta and the aortic valve closes at the maximum pressure, $P_{\text{sys}}$
- The blood pressure begins to fall as the blood flows through the arteries
- The rate of flowing of the blood depends on the resistance of a blood vessel
  - Viscosity of the blood
  - Length of the vessels
  - Radius of the blood vessels
Arterial Blood Pressure:

Introduction
Blood Pressure
Radioactive Decay
Solution of Linear Growth and Decay Models
Newton’s Law of Cooling
Solution of General Linear Model
Pollution in a Lake
Cardiac Cycle
Arterial Blood Pressure
Modeling Blood Pressure
Diagnosis with Model
Example of Athlete

Joseph M. Mahaffy, ⟨mahaffy@math.sdsu.edu⟩
Lecture Notes – Linear Differential Equations — (10/71)
Arterial Blood Pressure:

- The viscosity of the blood is relatively constant, except under diseased states like erythrocytemia (or when athletes take erythropoietin or EPO to overstimulate the production of red blood cells)
Arterial Blood Pressure:

- The viscosity of the blood is relatively constant, except under diseased states like erythrocytemia (or when athletes take erythropoietin or EPO to overstimulate the production of red blood cells).
- The length of the blood vessels are relatively constant, except for when conditions like pregnancy or amputation occur.
Arterial Blood Pressure:

- The viscosity of the blood is relatively constant, except under diseased states like erythrocytemia (or when athletes take erythropoietin or EPO to overstimulate the production of red blood cells).
- The length of the blood vessels are relatively constant, except for when conditions like pregnancy or amputation occur.
- The main factor that changes resistance of the blood flow is change in the radius.
Arterial Blood Pressure:

- The viscosity of the blood is relatively constant, except under diseased states like erythrocytemia (or when athletes take erythropoietin or EPO to overstimulate the production of red blood cells)
- The length of the blood vessels are relatively constant, except for when conditions like pregnancy or amputation occur
- The main factor that changes resistance of the blood flow is change in the radius
- Blood pressure becomes a valuable tool for detecting narrowing of the blood vessels by hypertension or atherosclerosis
Modeling Blood Pressure

Modeling Blood Pressure:
Modeling Blood Pressure:

Experimentally, it has been observed that systemic blood flow, $Q_s$, is proportional to the difference between the arterial and venous pressures $(P_a(t) - P_v(t))$ with the proportionality dependent on the resistance.
Modeling Blood Pressure:

- Experimentally, it has been observed that **systemic blood flow**, \( Q_s \), is proportional to the difference between the arterial and venous pressures \( (P_a(t) - P_v(t)) \) with the proportionality dependent on the resistance.

- If \( R_s \) is the systemic resistance (mm Hg/liter/min), then we have the following equation:

\[
Q_s(t) = \frac{1}{R_s} \ (P_a(t) - P_v(t))
\]
Modeling Blood Pressure:

- Experimentally, it has been observed that **systemic blood flow**, \( Q_s \), is proportional to the difference between the arterial and venous pressures \( (P_a(t) - P_v(t)) \) with the proportionality dependent on the resistance.

- If \( R_s \) is the systemic resistance (mm Hg/liter/min), then we have the following equation:

\[
Q_s(t) = \frac{1}{R_s} (P_a(t) - P_v(t))
\]

- To simplify the model, we take advantage of the fact that venous pressures are very low, so we approximate the systemic flow by the equation:

\[
Q_s(t) = \frac{1}{R_s} P_a(t)
\]
Compliance:
Compliance:

- **Compliance** is the stretchability of a vessel, which is a property that allows a vessel to change the volume in response to pressure changes.
Compliance:

- **Compliance** is the stretchability of a vessel, which is a property that allows a vessel to change the volume in response to pressure changes.
- The higher the compliance the easier it is for a vessel to expand in response to increased pressure.
Compliance:

- **Compliance** is the stretchability of a vessel, which is a property that allows a vessel to change the volume in response to pressure changes.
- The higher the compliance the easier it is for a vessel to expand in response to increased pressure.
- Resistance and compliance have a roughly inverse relationship.
Compliance:

- **Compliance** is the stretchability of a vessel, which is a property that allows a vessel to change the volume in response to pressure changes.
- The higher the compliance the easier it is for a vessel to expand in response to increased pressure.
- Resistance and compliance have a roughly inverse relationship.
- Experimentally, the arterial volume, \( V_a \), is roughly equal to the compliance, \( C_a \), times the arterial pressure.

\[
V_a(t) = C_a P_a(t)
\]
Differential Equation for Blood Flow:
Differential Equation for Blood Flow:

- The flow representing the change in the arterial volume is given by the difference between the rate of flow entering the aorta and the rate of flow from the aorta.
Differential Equation for Blood Flow:

- The flow representing the change in the arterial volume is given by the difference between the rate of flow entering the aorta and the rate of flow from the aorta.
- Since the aortic valve is closed during systole, no blood is entering the aorta.
Differential Equation for Blood Flow:

- The flow representing the change in the arterial volume is given by the difference between the rate of flow entering the aorta and the rate of flow from the aorta.
- Since the aortic valve is closed during systole, no blood is entering the aorta.
- The differential equation is

\[
\frac{dV_a(t)}{dt} = \text{flow rate in} - \text{flow rate out} = 0 - Q_s(t)
\]
Differential Equation for Blood Flow:

- The flow representing the change in the arterial volume is given by the difference between the rate of flow entering the aorta and the rate of flow from the aorta.
- Since the aortic valve is closed during systole, no blood is entering the aorta.
- The differential equation is

\[
\frac{dV_a(t)}{dt} = \text{flow rate in} - \text{flow rate out} = 0 - Q_s(t)
\]

- Thus,

\[
\frac{dV_a(t)}{dt} = -\frac{1}{R_s} P_a(t)
\]
Differential Equation for Blood Flow: Since

\[ V_a(t) = C_a P_a(t), \]

\[ \frac{dV_a(t)}{dt} = C_a \frac{dP_a(t)}{dt} \]
Differential Equation for Blood Flow: Since
\[ V_a(t) = C_a P_a(t), \]
\[ \frac{dV_a(t)}{dt} = C_a \frac{dP_a(t)}{dt} \]

This gives the initial value problem
\[ \frac{dP_a(t)}{dt} = -\frac{1}{C_a R_s} P_a(t) \quad \text{with} \quad P_a(0) = P_{sys} \]
Modeling Blood Pressure

**Differential Equation for Blood Flow:** Since
\[ V_a(t) = C_a P_a(t), \]
\[ \frac{dV_a(t)}{dt} = C_a \frac{dP_a(t)}{dt} \]

This gives the **initial value problem**

\[ \frac{dP_a(t)}{dt} = -\frac{1}{C_a R_s} P_a(t) \quad \text{with} \quad P_a(0) = P_{sys} \]

The **solution** is

\[ P_a(t) = P_{sys} e^{-\frac{t}{C_a R_s}} \quad \text{for} \quad t \in [0, T] \]
Diagnosis with Model: How can this model be used to provide a non-invasive method for estimating the physiological parameters for compliance, $C_a$, and resistance, $R_s$.
Diagnosis with Model: How can this model be used to provide a non-invasive method for estimating the physiological parameters for compliance, $C_a$, and resistance, $R_s$.

- Measurable physiological quantities are
Diagnosis with Model: How can this model be used to provide a non-invasive method for estimating the physiological parameters for compliance, \( C_a \), and resistance, \( R_s \)

- Measurable physiological quantities are
  - The heart rate or pulse, \( \frac{1}{T} \)
Diagnosis with Model: How can this model be used to provide a non-invasive method for estimating the physiological parameters for compliance, $C_a$, and resistance, $R_s$.

- Measurable physiological quantities are
  - The heart rate or pulse, $\frac{1}{T}$
  - Cardiac output, $Q$, using a doppler sonogram
Diagnosis with Model: How can this model be used to provide a non-invasive method for estimating the physiological parameters for compliance, $C_a$, and resistance, $R_s$.

- Measurable physiological quantities are
  - The heart rate or pulse, $\frac{1}{T}$
  - Cardiac output, $Q$, using a doppler sonogram
  - The systolic and diastolic pressures, $P_{sys}$ and $P_{dia}$
Diagnosis with Model: How can this model be used to provide a non-invasive method for estimating the physiological parameters for compliance, $C_a$, and resistance, $R_s$.

- Measurable physiological quantities are
  - The heart rate or pulse, $\frac{1}{T}$
  - Cardiac output, $Q$, using a doppler sonogram
  - The systolic and diastolic pressures, $P_{sys}$ and $P_{dia}$
- **Compliance** comes from the stroke volume, $V$,

$$V = V_{sys} - V_{dia} = C_a P_{sys} - C_a P_{dia}$$
**Diagnosis with Model:** How can this model be used to provide a non-invasive method for estimating the physiological parameters for compliance, $C_a$, and resistance, $R_s$.

- **Measurable physiological quantities are**
  - The heart rate or pulse, $\frac{1}{T}$
  - Cardiac output, $Q$, using a doppler sonogram
  - The systolic and diastolic pressures, $P_{sys}$ and $P_{dia}$

- **Compliance** comes from the stroke volume, $V$,

$$V = V_{sys} - V_{dia} = C_a P_{sys} - C_a P_{dia}$$

- But $V = QT$, so **compliance** satisfies

$$C_a = \frac{QT}{P_{sys} - P_{dia}}$$
Resistance: The model gives the diastolic pressure just before the next heart beat

\[ P_{dia} = P_{sys} e^{-\frac{T}{C_a R_s}} \]

Solve this equation for the resistance, \( R_s \)

\[ R_s = \frac{T}{C_a (\ln(P_{sys}) - \ln(P_{dia}))} \]
Resistance: The model gives the diastolic pressure just before the next heart beat

\[ P_{dia} = P_{sys} e^{-\frac{T}{C_a R_s}} \]

Solve this equation for the resistance, \( R_s \)

\[ R_s = \frac{T}{C_a (\ln(P_{sys}) - \ln(P_{dia}))} \]

- Normal Person
Diagnosis with Model

**Resistance:** The model gives the diastolic pressure just before the next heart beat

\[ P_{dia} = P_{sys} e^{-\frac{T}{C_a R_s}} \]

Solve this equation for the **resistance**, \( R_s \)

\[ R_s = \frac{T}{C_a (\ln(P_{sys}) - \ln(P_{dia}))} \]

- Normal Person
  - Pulse of approximately 70 beats/min (\( \frac{1}{T} \))
Resistance: The model gives the diastolic pressure just before the next heart beat

\[ P_{\text{dia}} = P_{\text{sys}} e^{-\frac{T}{C_a R_s}} \]

Solve this equation for the resistance, \( R_s \)

\[ R_s = \frac{T}{C_a (\ln(P_{\text{sys}}) - \ln(P_{\text{dia}}))} \]

- **Normal Person**
  - Pulse of approximately 70 beats/min (\( \frac{1}{T} \))
  - Cardiac output of \( Q = 5.6 \) (liters/min)


**Resistance:** The model gives the diastolic pressure just before the next heart beat

\[ P_{\text{dia}} = P_{\text{sys}} e^{-\frac{T}{C_a R_s}} \]

Solve this equation for the **resistance**, \( R_s \)

\[ R_s = \frac{T}{C_a (\ln(P_{\text{sys}}) - \ln(P_{\text{dia}}))} \]

- **Normal Person**
  - Pulse of approximately 70 beats/min (\( \frac{1}{T} \))
  - Cardiac output of \( Q = 5.6 \) (liters/min)
  - Systolic and diastolic pressures of \( P_{\text{sys}} = 120 \) mm Hg and \( P_{\text{dia}} = 80 \) mm Hg
Diagnosis with Model

**Resistance:** The model gives the diastolic pressure just before the next heart beat

\[ P_{dia} = P_{sys} e^{-\frac{T}{C_a R_s}} \]

Solve this equation for the resistance, \( R_s \)

\[ R_s = \frac{T}{C_a (\ln(P_{sys}) - \ln(P_{dia}))} \]

- Normal Person
  - Pulse of approximately 70 beats/min (\( \frac{1}{T} \))
  - Cardiac output of \( Q = 5.6 \) (liters/min)
  - Systolic and diastolic pressures of \( P_{sys} = 120 \) mm Hg and \( P_{dia} = 80 \) mm Hg

- Compute the compliance and resistance for a normal person

\[ C_a = 0.002 \text{ (liters/mm Hg)} \quad \text{and} \quad R_s = 17.6 \text{ (mm Hg/liter/min)} \]
Example of an Athlete: Consider a trained athlete considered in very good condition
Example of an Athlete: Consider a trained athlete considered in very good condition

- Suppose an athlete has
Example of an Athlete: Consider a trained athlete considered in very good condition

- Suppose an athlete has
  - A pulse of 60 beats/min (at rest)
Example of an Athlete: Consider a trained athlete considered in very good condition

- Suppose an athlete has
  - A pulse of 60 beats/min (at rest)
  - A blood pressure of 120/75
Example of an Athlete: Consider a trained athlete considered in very good condition

- Suppose an athlete has
  - A pulse of 60 beats/min (at rest)
  - A blood pressure of 120/75
  - A measured cardiac output of 6 liters/min
Example of an Athlete: Consider a trained athlete considered in very good condition

- Suppose an athlete has
  - A pulse of **60 beats/min** (at rest)
  - A blood pressure of **120/75**
  - A measured cardiac output of **6 liters/min**

- Find the **compliance**, $C_a$, and systemic **resistance**, $R_s$, of the arteries for this individual
Solution: From the formula, compliance, $C_a$

$$C_a = \frac{QT}{P_{sys} - P_{dia}} = \frac{6.0/60}{120 - 75} = 0.00222 \text{ (liters/mm Hg)}$$
Example of Athlete

**Solution:** From the formula, compliance, \( C_a \)

\[
C_a = \frac{QT}{P_{sys} - P_{dia}} = \frac{6.0/60}{120 - 75} = 0.00222 \text{ (liters/mm Hg)}
\]

This is slightly larger than for a normal person
Example of Athlete

Solution: From the formula, compliance, $C_a$

\[ C_a = \frac{QT}{P_{sys} - P_{dia}} = \frac{6.0/60}{120 - 75} = 0.00222 \text{ (liters/mm Hg)} \]

This is slightly larger than for a normal person.

The systemic resistance, $R_s$, satisfies

\[ R_s = \frac{T}{C_a (\ln(P_{sys}) - \ln(P_{dia}))} = \frac{1/60}{0.00222(\ln(120) - \ln(75))} = 15.96 \text{ (mm Hg/liter/min)} \]
Solution: From the formula, compliance, $C_a$

$$C_a = \frac{QT}{P_{sys} - P_{dia}} = \frac{6.0/60}{120 - 75} = 0.00222 \text{ (liters/mm Hg)}$$

This is slightly larger than for a normal person.

The systemic resistance, $R_s$, satisfies

$$R_s = \frac{T}{C_a (\ln(P_{sys}) - \ln(P_{dia}))}$$

$$= \frac{1/60}{0.00222(\ln(120) - \ln(75))} = 15.96 \text{ (mm Hg/liter/min)}$$

This is lower than for a normal person, which is what we would expect for someone in better condition.
Radioactive Decay

**Radioactive Decay**: Radioactive elements are important in many biological applications.
Radioactive Decay

Radioactive Decay: Radioactive elements are important in many biological applications

- $^3\text{H}$ (tritium) is used to tag certain DNA base pairs
Radioactive Decay: Radioactive elements are important in many biological applications

- $^3$H (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of *E. coli* that are unable to manufacture one particular DNA base
Radioactive Decay: Radioactive elements are important in many biological applications

- $^3$H (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of *E. coli* that are unable to manufacture one particular DNA base
  - Using antibiotics, one uses the radioactive signal to determine how much DNA is replicated under a particular set of experimental conditions
Radioactive Decay

Radioactive Decay: Radioactive elements are important in many biological applications

- $^3\text{H}$ (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of *E. coli* that are unable to manufacture one particular DNA base
  - Using antibiotics, one uses the radioactive signal to determine how much DNA is replicated under a particular set of experimental conditions
- Radioactive iodine is often used to detect or treat thyroid problems
Radioactive Decay: Radioactive elements are important in many biological applications

- $^3$H (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of *E. coli* that are unable to manufacture one particular DNA base
  - Using antibiotics, one uses the radioactive signal to determine how much DNA is replicated under a particular set of experimental conditions
- Radioactive iodine is often used to detect or treat thyroid problems
- Most experiments are run so that radioactive decay is not an issue
Radioactive Decay: Radioactive elements are important in many biological applications

- $^3$H (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of *E. coli* that are unable to manufacture one particular DNA base
  - Using antibiotics, one uses the radioactive signal to determine how much DNA is replicated under a particular set of experimental conditions
- Radioactive iodine is often used to detect or treat thyroid problems
- Most experiments are run so that radioactive decay is not an issue
  - $^3$H has a half-life of 12.5 yrs
  - $^{131}$I has a half-life of 8 days
Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens.
Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens

- A living organism is continually changing its carbon with the environment
Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens

- A living organism is continually changing its carbon with the environment
  - Plants directly absorb CO$_2$ from the atmosphere
Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens

- A living organism is continually changing its carbon with the environment
  - Plants directly absorb CO₂ from the atmosphere
  - Animals get their carbon either directly or indirectly from plants
Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens.

- A living organism is continually changing its carbon with the environment.
  - Plants directly absorb CO$_2$ from the atmosphere.
  - Animals get their carbon either directly or indirectly from plants.
- Gamma radiation that bombards the Earth keeps the ratio of $^{14}$C to $^{12}$C fairly constant in the atmospheric CO$_2$.
Carbon Radiodating: One important application of radioactive decay is the dating of biological specimens

- A living organism is continually changing its carbon with the environment
  - Plants directly absorb CO$_2$ from the atmosphere
  - Animals get their carbon either directly or indirectly from plants
- Gamma radiation that bombards the Earth keeps the ratio of $^{14}$C to $^{12}$C fairly constant in the atmospheric CO$_2$
- $^{14}$C stays at a constant concentration until the organism dies
Modeling Carbon Radiodating: Radioactive carbon, $^{14}\text{C}$, decays with a half-life of 5730 yr
 Modeling Carbon Radiodating: Radioactive carbon, $^{14}\text{C}$, decays with a half-life of 5730 yr

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon
Carbon Radiodating

**Modeling Carbon Radiodating:** Radioactive carbon, $^{14}$C, decays with a **half-life of 5730 yr**

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon
- The loss of $^{14}$C from a sample at any time $t$ is proportional to the amount of $^{14}$C remaining
Carbon Radiodating

Modeling Carbon Radiodating: Radioactive carbon, $^{14}$C, decays with a half-life of 5730 yr

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon
- The loss of $^{14}$C from a sample at any time $t$ is proportional to the amount of $^{14}$C remaining
- Let $R(t)$ be the dpm per gram of $^{14}$C from an ancient object
Carbon Radiodating

Modeling Carbon Radiodating: Radioactive carbon, $^{14}$C, decays with a half-life of 5730 yr

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon
- The loss of $^{14}$C from a sample at any time $t$ is proportional to the amount of $^{14}$C remaining
- Let $R(t)$ be the dpm per gram of $^{14}$C from an ancient object
- The differential equation for a gram of $^{14}$C

$$\frac{dR(t)}{dt} = -kR(t) \quad \text{with} \quad R(0) = 15.3$$
Carbon Radiodating

Modeling Carbon Radiodating: Radioactive carbon, $^{14}\text{C}$, decays with a half-life of 5730 yr

- Living tissue shows a radioactivity of about 15.3 disintegrations per minute (dpm) per gram of carbon
- The loss of $^{14}\text{C}$ from a sample at any time $t$ is proportional to the amount of $^{14}\text{C}$ remaining
- Let $R(t)$ be the dpm per gram of $^{14}\text{C}$ from an ancient object
- The differential equation for a gram of $^{14}\text{C}$

$$\frac{dR(t)}{dt} = -kR(t) \quad \text{with} \quad R(0) = 15.3$$

- This differential equation has the solution

$$R(t) = 15.3 e^{-kt}, \quad \text{where} \quad k = \frac{\ln(2)}{5730} = 0.000121$$
Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon
Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon. Find the age of this object.
Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon

Find the age of this object

Solution: From above

\[ 5.2 = 15.3 e^{-kt} \]
Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon. Find the age of this object.

Solution: From above

\[
\begin{align*}
5.2 &= 15.3 e^{-kt} \\
\frac{e^{kt}}{5.2} &= \frac{15.3}{5.2} = 2.94
\end{align*}
\]
Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon.

Find the age of this object.

Solution: From above

\[
5.2 = 15.3 e^{-kt}
\]
\[
e^{kt} = \frac{15.3}{5.2} = 2.94
\]
\[
kt = \ln(2.94)
\]
Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon.

Find the age of this object.

Solution: From above

\[
5.2 = 15.3 e^{-kt}
\]

\[
e^{kt} = \frac{15.3}{5.2} = 2.94
\]

\[
kt = \ln(2.94)
\]

Thus, 

\[
t = \frac{\ln(2.94)}{k} = 8915 \text{ yr},
\]
Example: Carbon Radiodating

Example Carbon Radiodating: Suppose that an object is found to have a radioactive count of 5.2 dpm per g of carbon. Find the age of this object.

Solution: From above

\[ 5.2 = 15.3 e^{-kt} \]
\[ e^{kt} = \frac{15.3}{5.2} = 2.94 \]
\[ kt = \ln(2.94) \]

Thus, \( t = \frac{\ln(2.94)}{k} = 8915 \text{ yr} \), so the object is about 9000 yrs old.
Hyperthyroidism is a serious health problem caused by an overactive thyroid.
Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is **thyroxine**, which stimulates the release of other hormones
Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is **thyroxine**, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is thyroxine, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
- Result is low blood sugar causing lethargy or mood disorders and sexual dysfunction
Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is thyroxine, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
- Result is low blood sugar causing lethargy or mood disorders and sexual dysfunction
- One treatment for hyperthyroidism is ablating the thyroid with a large dose of radioactive iodine, $^{131}\text{I}$
Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is thyroxine, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
- Result is low blood sugar causing lethargy or mood disorders and sexual dysfunction
- One treatment for hyperthyroidism is ablating the thyroid with a large dose of radioactive iodine, $^{131}$I
  - The thyroid concentrates iodine brought into the body
Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is thyroxine, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
- Result is low blood sugar causing lethargy or mood disorders and sexual dysfunction
- One treatment for hyperthyroidism is ablating the thyroid with a large dose of radioactive iodine, $^{131}\text{I}$
  - The thyroid concentrates iodine brought into the body
  - The $^{131}\text{I}$ undergoes both $\beta$ and $\gamma$ radioactive decay, which destroys tissue
Hyperthyroidism is a serious health problem caused by an overactive thyroid

- The primary hormone released is thyroxine, which stimulates the release of other hormones
- Too many other hormones, such as insulin and the sex hormones
- Result is low blood sugar causing lethargy or mood disorders and sexual dysfunction
- One treatment for hyperthyroidism is ablating the thyroid with a large dose of radioactive iodine, $^{131}$I
  - The thyroid concentrates iodine brought into the body
  - The $^{131}$I undergoes both $\beta$ and $\gamma$ radioactive decay, which destroys tissue
  - Patient is given medicine to supplement the loss of thyroxine
Hyperthyroidism

Hyperthyroidism: Treatment
Hyperthyroidism: Treatment

Based upon the thyroid condition and body mass, a standard dose ranges from 110-150 mCi (milliCuries), given in a special “cocktail”
Hyperthyroidism: Treatment

- Based upon the thyroid condition and body mass, a standard dose ranges from 110-150 mCi (milliCuries), given in a special “cocktail”
- It is assumed that almost 100% of the $^{131}\text{I}$ is absorbed by the blood from the gut
Hyperthyroidism: Treatment

- Based upon the thyroid condition and body mass, a standard dose ranges from **110-150 mCi** (milliCuries), given in a special “cocktail”

- It is assumed that almost **100%** of the **$^{131}$I** is absorbed by the blood from the gut

- The thyroid uptakes **30%** of this isotope of iodine, peaking around 3 days
Hyperthyroidism: Treatment

- Based upon the thyroid condition and body mass, a standard dose ranges from **110-150 mCi** (milliCuries), given in a special “cocktail”
- It is assumed that almost **100%** of the $^{131}\text{I}$ is absorbed by the blood from the gut
- The thyroid uptakes **30%** of this isotope of iodine, peaking around 3 days
- The remainder is excreted in the urine
Hyperthyroidism: Treatment

- Based upon the thyroid condition and body mass, a standard dose ranges from **110-150 mCi** (milliCuries), given in a special “cocktail”
- It is assumed that almost **100%** of the $^{131}\text{I}$ is absorbed by the blood from the gut
- The thyroid uptakes **30%** of this isotope of iodine, peaking around 3 days
- The remainder is excreted in the urine
- The half-life of $^{131}\text{I}$ is **8 days**, so this isotope rapidly decays
Hyperthyroidism: Treatment

- Based upon the thyroid condition and body mass, a standard dose ranges from 110-150 mCi (milliCuries), given in a special “cocktail”
- It is assumed that almost 100% of the $^{131}$I is absorbed by the blood from the gut
- The thyroid uptakes 30% of this isotope of iodine, peaking around 3 days
- The remainder is excreted in the urine
- The half-life of $^{131}$I is 8 days, so this isotope rapidly decays
- Still the patient must remain in a designated room for 3-4 days for this procedure, so that he or she does not irradiate the public from his or her treatment
Hyperthyroidism Example: Assume that a patient is given a 120 mCi cocktail of \(^{131}\text{I}\) and that 30\% is absorbed by the thyroid
Hyperthyroidism Example: Assume that a patient is given a 120 mCi cocktail of $^{131}$I and that 30% is absorbed by the thyroid.

- Find the amount of $^{131}$I in the thyroid (in mCi), if the patient is released four days after swallowing the radioactive cocktail.
Hyperthyroidism Example: Assume that a patient is given a **120 mCi** cocktail of **$^{131}$I** and that **30%** is absorbed by the thyroid

- Find the amount of **$^{131}$I** in the thyroid (in mCi), if the patient is released four days after swallowing the radioactive cocktail
- Calculate how many mCis the patient’s thyroid retains after 30 days, assuming that it was taken up by the thyroid and not excreted in the urine
Hyperthyroidism

Solution:
Hyperthyroidism

Solution:

Assume for simplicity of the model that the $^{131}\text{I}$ is immediately absorbed into the thyroid, then stays there until it undergoes radioactive decay.
Solution:

- Assume for simplicity of the model that the $^{131}$I is immediately absorbed into the thyroid, then stays there until it undergoes radioactive decay.

- Since the thyroid uptakes 30% of the 120 mCi, assume that the thyroid has 36 mCi immediately after the procedure.
Hyperthyroidism

Solution:

- Assume for simplicity of the model that the $^{131}$I is immediately absorbed into the thyroid, then stays there until it undergoes radioactive decay.
- Since the thyroid uptakes 30% of the 120 mCi, assume that the thyroid has 36 mCi immediately after the procedure.
- This is an oversimplification as it takes time for the $^{131}$I to accumulate in the thyroid.
Hyperthyroidism

Solution:

- Assume for simplicity of the model that the $^{131}$I is immediately absorbed into the thyroid, then stays there until it undergoes radioactive decay.
- Since the thyroid uptakes 30% of the 120 mCi, assume that the thyroid has 36 mCi immediately after the procedure.
- This is an oversimplification as it takes time for the $^{131}$I to accumulate in the thyroid.
- This allows the simple model

$$\frac{dR}{dt} = -k \, R(t) \quad \text{with} \quad R(0) = 36 \, \text{mCi}$$
Solution (cont): The radioactive decay model is

\[
\frac{dR}{dt} = -k R(t) \quad \text{with} \quad R(0) = 36 \text{ mCi}
\]
Solution (cont): The radioactive decay model is

\[
\frac{dR}{dt} = -k R(t) \quad \text{with} \quad R(0) = 36 \text{ mCi}
\]

The solution is

\[
R(t) = 36 e^{-kt}
\]
Hyperthyroidism

Solution (cont): The radioactive decay model is

\[
\frac{dR}{dt} = -kR(t) \quad \text{with} \quad R(0) = 36 \text{ mCi}
\]

- The solution is

\[
R(t) = 36 e^{-kt}
\]

- Since the half-life of \(^{131}\text{I}\) is 8 days, after 8 days there will be 18 mCi of \(^{131}\text{I}\)
Solution (cont): The radioactive decay model is

\[ \frac{dR}{dt} = -kR(t) \quad \text{with} \quad R(0) = 36 \text{ mCi} \]

- The solution is
  \[ R(t) = 36 e^{-kt} \]
- Since the half-life of $^{131}\text{I}$ is 8 days, after 8 days there will be 18 mCi of $^{131}\text{I}$
- Thus, $R(8) = 18 = 36 e^{-8k}$, so
  \[ e^{8k} = 2 \quad \text{or} \quad 8k = \ln(2) \]
Hyperthyroidism

Solution (cont): The radioactive decay model is

\[ \frac{dR}{dt} = -k R(t) \quad \text{with} \quad R(0) = 36 \text{ mCi} \]

- The solution is
  \[ R(t) = 36 e^{-kt} \]

- Since the half-life of $^{131}\text{I}$ is 8 days, after 8 days there will are 18 mCi of $^{131}\text{I}$
- Thus, $R(8) = 18 = 36 e^{-8k}$, so
  \[ e^{8k} = 2 \quad \text{or} \quad 8k = \ln(2) \]
- Thus, $k = \frac{\ln(2)}{8} = 0.0866 \text{ day}^{-1}$
Solution (cont): Since

\[ R(t) = 36 e^{-kt} \quad \text{with} \quad k = 0.0866 \text{ day}^{-1} \]
Hyperthyroidism

Solution (cont): Since

\[ R(t) = 36 e^{-kt} \quad \text{with} \quad k = 0.0866 \text{ day}^{-1} \]

- At the time of the patient’s release \( t = 4 \) days, so in the thyroid

\[ R(4) = 36 e^{-4k} = \frac{36}{\sqrt{2}} = 25.46 \text{ mCi} \]

Joseph M. Mahaffy, \{mahaffy@math.sdsu.edu\}
Hyperthyroidism

Solution (cont): Since

$$R(t) = 36 e^{-kt}$$

with $$k = 0.0866 \text{ day}^{-1}$$

- At the time of the patient’s release $$t = 4$$ days, so in the thyroid

$$R(4) = 36 e^{-4k} = \frac{36}{\sqrt{2}} = 25.46 \text{ mCi}$$

- After 30 days, we find in the thyroid

$$R(30) = 36 e^{-30k} = 2.68 \text{ mCi}$$
Graph of $R(t)$

![Graph showing the remaining in thyroid over time, with a line indicating the half-life.](image)
General Solution to Linear Growth and Decay Models:

Consider

\[ \frac{dy}{dt} = ay \quad \text{with} \quad y(t_0) = y_0 \]
General Solution to Linear Growth and Decay Models:
Consider
\[
\frac{dy}{dt} = ay \quad \text{with} \quad y(t_0) = y_0
\]

The solution is
\[
y(t) = y_0 e^{a(t-t_0)}
\]
Example: Linear Decay Model: Consider

\[ \frac{dy}{dt} = -0.3 y \quad \text{with} \quad y(4) = 12 \]
Example: Linear Decay Model: Consider

\[ \frac{dy}{dt} = -0.3 \, y \quad \text{with} \quad y(4) = 12 \]

The solution is

\[ y(t) = 12 \, e^{-0.3(t-4)} \]
Newton’s Law of Cooling:

Joseph M. Mahaffy, (mahaffy@math.sdsu.edu)
Newton’s Law of Cooling:

- After a murder (or death by other causes), the forensic scientist takes the temperature of the body
Newton’s Law of Cooling:

- After a murder (or death by other causes), the forensic scientist takes the temperature of the body.
- Later the temperature of the body is taken again to find the rate at which the body is cooling.
Newton’s Law of Cooling:

- After a murder (or death by other causes), the forensic scientist takes the temperature of the body.
- Later the temperature of the body is taken again to find the rate at which the body is cooling.
- Two (or more) data points are used to extrapolate back to when the murder occurred.
Newton’s Law of Cooling:

- After a murder (or death by other causes), the forensic scientist takes the temperature of the body.
- Later the temperature of the body is taken again to find the rate at which the body is cooling.
- Two (or more) data points are used to extrapolate back to when the murder occurred.
- This property is known as Newton’s Law of Cooling.
Newton’s Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature.
Newton’s Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature.

- If $T(t)$ is the temperature of the body, then it satisfies the differential equation

\[
\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0
\]
Newton’s Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature.

- If $T(t)$ is the temperature of the body, then it satisfies the differential equation

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0$$

- The parameter $k$ is dependent on the specific properties of the particular object (body in this case).
Newton’s Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature.

- If $T(t)$ is the temperature of the body, then it satisfies the differential equation

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0$$

- The parameter $k$ is dependent on the specific properties of the particular object (body in this case)
- $T_e$ is the environmental temperature
Newton’s Law of Cooling states that the rate of change in temperature of a cooling body is proportional to the difference between the temperature of the body and the surrounding environmental temperature.

- If $T(t)$ is the temperature of the body, then it satisfies the differential equation

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T(0) = T_0$$

- The parameter $k$ is dependent on the specific properties of the particular object (body in this case)
- $T_e$ is the environmental temperature
- $T_0$ is the initial temperature of the object
Murder Example

Murder Example
Murder Example

Suppose that a murder victim is found at 8:30 am
Murder Example

Suppose that a murder victim is found at 8:30 am
The temperature of the body at that time is 30°C
Murder Example

Suppose that a murder victim is found at 8:30 am

The temperature of the body at that time is 30°C

Assume that the room in which the murder victim lay was a constant 22°C
Murder Example

Suppose that a murder victim is found at 8:30 am
The temperature of the body at that time is 30°C
Assume that the room in which the murder victim lay was a constant 22°C
Suppose that an hour later the temperature of the body is 28°C
Murder Example

Suppose that a murder victim is found at 8:30 am.
The temperature of the body at that time is 30°C.
Assume that the room in which the murder victim lay was
a constant 22°C.
Suppose that an hour later the temperature of the body is
28°C.
Normal temperature of a human body when it is alive is
37°C.
**Murder Example**

- Suppose that a murder victim is found at 8:30 am
- The temperature of the body at that time is 30°C
- Assume that the room in which the murder victim lay was a constant 22°C
- Suppose that an hour later the temperature of the body is 28°C
- Normal temperature of a human body when it is alive is 37°C
- Use this information to determine the approximate time that the murder occurred
**Solution:** From the model for Newton’s Law of Cooling and the information that is given, if we set $t = 0$ to be 8:30 am, then we solve the initial value problem

\[
\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30
\]
**Solution:** From the model for Newton’s Law of Cooling and the information that is given, if we set $t = 0$ to be 8:30 am, then we solve the initial value problem

$$\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30$$

- Make a change of variables $z(t) = T(t) - 22$
Solution: From the model for Newton’s Law of Cooling and the information that is given, if we set $t = 0$ to be 8:30 am, then we solve the initial value problem

$$\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30$$

- Make a change of variables $z(t) = T(t) - 22$
- Then $z'(t) = T'(t)$, so the differential equation above becomes

$$\frac{dz}{dt} = -kz(t), \quad \text{with} \quad z(0) = T(0) - 22 = 8$$
Solution: From the model for Newton’s Law of Cooling and the information that is given, if we set $t = 0$ to be 8:30 am, then we solve the initial value problem

$$\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30$$

- Make a change of variables $z(t) = T(t) - 22$
- Then $z'(t) = T'(t)$, so the differential equation above becomes

$$\frac{dz}{dt} = -kz(t), \quad \text{with} \quad z(0) = T(0) - 22 = 8$$

- This is the radioactive decay problem that we solved
Murder Example

Solution: From the model for Newton’s Law of Cooling and the information that is given, if we set $t = 0$ to be 8:30 am, then we solve the initial value problem

$$\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30$$

- Make a change of variables $z(t) = T(t) - 22$
- Then $z'(t) = T'(t)$, so the differential equation above becomes

$$\frac{dz}{dt} = -kz(t), \quad \text{with} \quad z(0) = T(0) - 22 = 8$$

- This is the radioactive decay problem that we solved
- The solution is

$$z(t) = 8e^{-kt}$$
Solution (cont): From the solution \( z(t) = 8e^{-kt} \), we have

\[
\begin{align*}
z(t) &= T(t) - 22, \quad \text{so} \quad T(t) = z(t) + 22 \\
T(t) &= 22 + 8e^{-kt}
\end{align*}
\]
Solution (cont): From the solution $z(t) = 8e^{-kt}$, we have

$$z(t) = T(t) - 22,$$

so

$$T(t) = z(t) + 22$$

$$T(t) = 22 + 8e^{-kt}$$

- One hour later the body temperature is $28^\circ C$

$$T(1) = 28 = 22 + 8e^{-k}$$
Murder Example

Solution (cont): From the solution \( z(t) = 8e^{-kt} \), we have

\[
\begin{align*}
  z(t) &= T(t) - 22, \quad \text{so} \quad T(t) = z(t) + 22 \\
  T(t) &= 22 + 8e^{-kt}
\end{align*}
\]

- One hour later the body temperature is 28°C

\[
T(1) = 28 = 22 + 8e^{-k}
\]

- Solving

\[
6 = 8e^{-k} \quad \text{or} \quad e^k = \frac{4}{3}
\]
Solution (cont): From the solution $z(t) = 8e^{-kt}$, we have

\[
\begin{align*}
z(t) &= T(t) - 22, \quad \text{so} \quad T(t) = z(t) + 22 \\
T(t) &= 22 + 8e^{-kt}
\end{align*}
\]

- One hour later the body temperature is $28^\circ C$

\[
T(1) = 28 = 22 + 8e^{-k}
\]

- Solving

\[
6 = 8e^{-k} \quad \text{or} \quad e^k = \frac{4}{3}
\]

- Thus, $k = \ln\left(\frac{4}{3}\right) = 0.2877$
Solution (cont): It only remains to find out when the murder occurred
Murder Example

Solution (cont): It only remains to find out when the murder occurred

- At the time of death, $t_d$, the body temperature is $37^\circ C$

$$T(t_d) = 37 = 22 + 8e^{-k}$$
Murder Example

Solution (cont): It only remains to find out when the murder occurred

- At the time of death, \( t_d \), the body temperature is 37°C

\[
T(t_d) = 37 = 22 + 8e^{-k}
\]

- Thus,

\[
8e^{-kt_d} = 37 - 22 = 15 \quad \text{or} \quad e^{-kt_d} = \frac{15}{8} = 1.875
\]
Solution (cont): It only remains to find out when the murder occurred

- At the time of death, \( t_d \), the body temperature is \( 37 \degree C \)
  \[
  T(t_d) = 37 = 22 + 8 e^{-k}
  \]
- Thus,
  \[
  8 e^{-k t_d} = 37 - 22 = 15 \quad \text{or} \quad e^{-k t_d} = \frac{15}{8} = 1.875
  \]
- This gives \( -k t_d = \ln(1.875) \) or
  \[
  t_d = -\frac{\ln(1.875)}{k} = -2.19
  \]
Solution (cont): It only remains to find out when the murder occurred.

- At the time of death, \( t_d \), the body temperature is 37°C
  \[
  T(t_d) = 37 = 22 + 8 e^{-k}
  \]
- Thus,
  \[
  8 e^{-kt_d} = 37 - 22 = 15 \quad \text{or} \quad e^{-kt_d} = \frac{15}{8} = 1.875
  \]
- This gives \(-kt_d = \ln(1.875)\) or
  \[
  t_d = -\frac{\ln(1.875)}{k} = -2.19
  \]
- The murder occurred about 2 hours 11 minutes before the body was found, which places the time of death around 6:19 am.

Joseph M. Mahaffy, (mahaffy@math.sdsu.edu)
Cooling Tea: We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk.
Cooling Tea: We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk.

- Begin with \( \frac{4}{5} \) cup of boiling hot tea, \( T(0) = 100^\circ C \)
Cooling Tea: We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk.

- Begin with $\frac{4}{5}$ cup of boiling hot tea, $T(0) = 100^\circ C$
- Assume the tea cools according to Newton’s law of cooling

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T_e = 20^\circ C$$
Cooling Tea: We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk

- Begin with $\frac{4}{5}$ cup of boiling hot tea, $T(0) = 100^\circ C$
- Assume the tea cools according to Newton’s law of cooling

\[
\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T_e = 20^\circ C
\]

- $k$ is the cooling constant based on the properties of the cup to be calculated
Cooling Tea: We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk.

- Begin with $\frac{4}{5}$ cup of boiling hot tea, $T(0) = 100^\circ C$
- Assume the tea cools according to Newton’s law of cooling

$$\frac{dT}{dt} = -k(T(t) - T_e) \quad \text{with} \quad T_e = 20^\circ C$$

- $k$ is the cooling constant based on the properties of the cup to be calculated.

- a. In the first scenario, you let the tea cool for 5 minutes, then add $\frac{1}{5}$ cup of cold milk, $5^\circ C$
Cooling Tea (cont):
Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
- Determine the value of \( k \), which we assume stays the same in this problem
Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
- Determine the value of $k$, which we assume stays the same in this problem
- Mix in the milk, assuming that the temperature mixes perfectly in proportion to the volume of the two liquids
Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
- Determine the value of $k$, which we assume stays the same in this problem
- Mix in the milk, assuming that the temperature mixes perfectly in proportion to the volume of the two liquids
- b. In the second case, add $\frac{1}{5}$ cup of cold milk, 5°C, immediately and mix it thoroughly
Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
- Determine the value of $k$, which we assume stays the same in this problem
- Mix in the milk, assuming that the temperature mixes perfectly in proportion to the volume of the two liquids
- b. In the second case, add $\frac{1}{5}$ cup of cold milk, 5°C, immediately and mix it thoroughly
- Find how long until each cup of tea reaches a temperature of 70°C
Solution of Cooling Tea: Find the rate constant $k$ for

$$\frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 100 \quad \text{and} \quad T(2) = 95$$
Solution of Cooling Tea: Find the rate constant $k$ for

$$\frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 100 \quad \text{and} \quad T(2) = 95$$

- Let $z(t) = T(t) - 20$, so $z(0) - T(0) - 20 = 80$
Solution of Cooling Tea: Find the rate constant $k$ for

$$\frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 100 \quad \text{and} \quad T(2) = 95$$

- Let $z(t) = T(t) - 20$, so $z(0) - T(0) - 20 = 80$
- Since $z'(t) = T'(t)$, the initial value problem becomes

$$\frac{dz}{dt} = -k z(t), \quad z(0) = 80$$
Solution of Cooling Tea: Find the rate constant $k$ for

$$\frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 100 \quad \text{and} \quad T(2) = 95$$

- Let $z(t) = T(t) - 20$, so $z(0) - T(0) - 20 = 80$
- Since $z'(t) = T'(t)$, the initial value problem becomes

$$\frac{dz}{dt} = -k z(t), \quad z(0) = 80$$

- The solution is

$$z(t) = 80 e^{-kt} = T(t) - 20$$
Cooling Tea

Solution of Cooling Tea: Find the rate constant $k$ for

$$\frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 100 \quad \text{and} \quad T(2) = 95$$

- Let $z(t) = T(t) - 20$, so $z(0) - T(0) - 20 = 80$
- Since $z'(t) = T'(t)$, the initial value problem becomes

$$\frac{dz}{dt} = -k z(t), \quad z(0) = 80$$

- The solution is

$$z(t) = 80 e^{-kt} = T(t) - 20$$

- Thus,

$$T(t) = 80 e^{-kt} + 20$$
Solution (cont): The solution is

\[ T(t) = 80 e^{-kt} + 20 \]
Solution (cont): The solution is

\[ T(t) = 80 e^{-kt} + 20 \]

- Since \( T(2) = 95 \),

\[ 95 = 80e^{-2k} + 20 \quad \text{or} \quad e^{2k} = \frac{80}{75} \]
Solution (cont): The solution is

\[ T(t) = 80e^{-kt} + 20 \]

- Since \( T(2) = 95 \),

\[ 95 = 80e^{-2k} + 20 \quad \text{or} \quad e^{2k} = \frac{80}{75} \]

\[ k = \frac{\ln\left(\frac{80}{75}\right)}{2} = 0.03227 \]
Solution (cont): The solution is

\[ T(t) = 80 e^{-kt} + 20 \]

- Since \( T(2) = 95 \),

\[ 95 = 80e^{-2k} + 20 \quad \text{or} \quad e^{2k} = \frac{80}{75} \]

- \( k = \frac{\ln\left(\frac{80}{75}\right)}{2} = 0.03227 \)

- Find the temperature at 5 min

\[ T(5) = 80e^{-5k} + 20 = 88.1^\circ C \]
Solution (cont): The solution is

\[ T(t) = 80e^{-kt} + 20 \]

- Since \( T(2) = 95 \),

\[ 95 = 80e^{-2k} + 20 \quad \text{or} \quad e^{2k} = \frac{80}{75} \]

\[ k = \frac{\ln\left(\frac{80}{75}\right)}{2} = 0.03227 \]

- Find the temperature at 5 min

\[ T(5) = 80e^{-5k} + 20 = 88.1^\circ C \]

- Now mix the \( \frac{4}{5} \) cup of tea at 88.1°C with the \( \frac{1}{5} \) cup of milk at 5°C, so

\[ T_+(5) = 88.1 \left(\frac{4}{5}\right) + \left(5\frac{1}{5}\right) = 71.5^\circ C \]
Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

\[ T_+(5) = 71.5^\circ C \]
Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

\[ T_+ (5) = 71.5 ^\circ C \]

- The new initial value problem is

\[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(5) = 71.5 ^\circ C \]
Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

\[ T_+(5) = 71.5^\circ C \]

- The new initial value problem is

\[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(5) = 71.5^\circ C \]

- With the same substitution, \( z(t) = T(t) - 20 \),

\[ \frac{dz}{dt} = -kz, \quad z(5) = 51.5 \]
Cooling Tea

**Solution (cont):** For the first scenario, the temperature after adding the milk after 5 min satisfies 

\[ T_+(5) = 71.5^\circ C \]

- The new initial value problem is

\[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(5) = 71.5^\circ C \]

- With the same substitution, \( z(t) = T(t) - 20 \),

\[ \frac{dz}{dt} = -kz, \quad z(5) = 51.5 \]

- This has the solution

\[ z(t) = 51.5e^{-k(t-5)} = T(t) - 20 \]
Solution (cont): For the first scenario, the temperature satisfies

\[ T(t) = 51.5e^{-k(t-5)} + 20 \]
Solution (cont): For the first scenario, the temperature satisfies

\[ T(t) = 51.5e^{-k(t-5)} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 51.5e^{-k(t-5)} + 20 \]
Solution (cont): For the first scenario, the temperature satisfies

\[ T(t) = 51.5e^{-k(t-5)} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 51.5e^{-k(t-5)} + 20 \]

- Thus,

\[ e^{k(t-5)} = \frac{51.5}{50} \]
Solution (cont): For the first scenario, the temperature satisfies

\[ T(t) = 51.5e^{-k(t-5)} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 51.5e^{-k(t-5)} + 20 \]

- Thus,

\[ e^{k(t-5)} = \frac{51.5}{50} \]

- It follows that \( k(t - 5) = \ln(51.5/50) \), so

\[ t = 5 + \frac{\ln(51.5/50)}{k} = 5.92 \text{ min} \]
Solution (cont): For the second scenario, we mix the tea and milk, so

\[ T(0) = 100 \left( \frac{4}{5} \right) + 5 \left( \frac{1}{5} \right) = 81^\circ C \]
Solution (cont): For the second scenario, we mix the tea and milk, so

\[ T(0) = 100 \left( \frac{4}{5} \right) + 5 \left( \frac{1}{5} \right) = 81^\circ C \]

- The new initial value problem is

\[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 81^\circ C \]
Solution (cont): For the second scenario, we mix the tea and milk, so

\[ T(0) = 100 \left( \frac{4}{5} \right) + 5 \left( \frac{1}{5} \right) = 81^\circ \text{C} \]

- The new initial value problem is

\[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 81^\circ \text{C} \]

- With \( z(t) = T(t) - 20 \),

\[ \frac{dz}{dt} = -kz(t), \quad z(0) = 61 \]
Solution (cont): For the second scenario, we mix the tea and milk, so

\[ T(0) = 100 \left( \frac{4}{5} \right) + 5 \left( \frac{1}{5} \right) = 81^\circ C \]

- The new initial value problem is

\[ \frac{dT}{dt} = -k(T(t) - 20), \quad T(0) = 81^\circ C \]

- With \( z(t) = T(t) - 20 \),

\[ \frac{dz}{dt} = -k z(t), \quad z(0) = 61 \]

- This has the solution

\[ z(t) = 61e^{-kt} = T(t) - 20 \]
Solution (cont): For the second scenario, the solution is

\[ T(t) = 61 e^{-kt} + 20 \]
Solution (cont): For the second scenario, the solution is

\[ T(t) = 61e^{-kt} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 61e^{-kt} + 20 \]
Solution (cont): For the second scenario, the solution is

\[ T(t) = 61 e^{-kt} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 61e^{-kt} + 20 \]

- Thus,

\[ e^{kt} = \frac{61}{50} \]
Solution (cont): For the second scenario, the solution is

\[ T(t) = 61 e^{-kt} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 61e^{-kt} + 20 \]

- Thus,

\[ e^{kt} = \frac{61}{50} \]

- Since \( kt = \ln \left( \frac{61}{50} \right) \),

\[ t = \frac{\ln(61/50)}{k} = 6.16 \text{ min} \]
Solution (cont): For the second scenario, the solution is

\[ T(t) = 61 e^{-kt} + 20 \]

- To find when the tea is 70°C, solve

\[ 70 = 61e^{-kt} + 20 \]

- Thus,

\[ e^{kt} = \frac{61}{50} \]

- Since \( kt = \ln \left( \frac{61}{50} \right) \),

\[ t = \frac{\ln(61/50)}{k} = 6.16 \text{ min} \]

- Waiting to pour in the milk for 5 minutes, saves about 15 seconds in cooling time
Graph of Cooling Tea
Consider the Linear Model

\[ \frac{dy}{dt} = ay + b \hspace{1cm} \text{with} \hspace{1cm} y(t_0) = y_0 \]
Solution of General Linear Model

Consider the Linear Model

\[
\frac{dy}{dt} = ay + b \quad \text{with} \quad y(t_0) = y_0
\]

Rewrite equation as

\[
\frac{dy}{dt} = a \left( y + \frac{b}{a} \right)
\]
Solution of General Linear Model

Consider the Linear Model

\[ \frac{dy}{dt} = ay + b \quad \text{with} \quad y(t_0) = y_0 \]

Rewrite equation as

\[ \frac{dy}{dt} = a \left( y + \frac{b}{a} \right) \]

Make the substitution \( z(t) = y(t) + \frac{b}{a} \), so

\[ \frac{dz}{dt} = \frac{dy}{dt} \quad \text{and} \quad z(t_0) = y_0 + \frac{b}{a} \]

\[ \frac{dz}{dt} = az \quad \text{with} \quad z(t_0) = y_0 + \frac{b}{a} \]
Solution of General Linear Model

The shifted model is

\[
\frac{dz}{dt} = az \quad \text{with} \quad z(t_0) = y_0 + \frac{b}{a}
\]
Solution of General Linear Model

The shifted model is

\[
\frac{dz}{dt} = az \quad \text{with} \quad z(t_0) = y_0 + \frac{b}{a}
\]

The solution to this problem is

\[
z(t) = \left( y_0 + \frac{b}{a} \right) e^{a(t-t_0)} = y(t) + \frac{b}{a}
\]
Solution of General Linear Model

The shifted model is

\[
\frac{dz}{dt} = az \quad \text{with} \quad z(t_0) = y_0 + \frac{b}{a}
\]

The solution to this problem is

\[
z(t) = \left( y_0 + \frac{b}{a} \right) e^{a(t-t_0)} = y(t) + \frac{b}{a}
\]

The solution is

\[
y(t) = \left( y_0 + \frac{b}{a} \right) e^{a(t-t_0)} - \frac{b}{a}
\]
Example of Linear Model

Consider the Linear Model

\[ \frac{dy}{dt} = 5 - 0.2y \quad \text{with} \quad y(3) = 7 \]
Example of Linear Model Consider the Linear Model

\[ \frac{dy}{dt} = 5 - 0.2y \quad \text{with} \quad y(3) = 7 \]

Rewrite equation as

\[ \frac{dy}{dt} = -0.2(y - 25) \]
Example of Linear Model

Consider the Linear Model

\[ \frac{dy}{dt} = 5 - 0.2y \quad \text{with} \quad y(3) = 7 \]

Rewrite equation as

\[ \frac{dy}{dt} = -0.2(y - 25) \]

Make the substitution \( z(t) = y(t) - 25 \), so \( \frac{dz}{dt} = \frac{dy}{dt} \) and \( z(3) = -18 \)

\[ \frac{dz}{dt} = -0.2z \quad \text{with} \quad z(3) = -18 \]
Example of Linear Model The substituted model is

\[
\frac{dz}{dt} = -0.2 \, z \quad \text{with} \quad z(3) = -18
\]
Example of Linear Model

The substituted model is

\[
\frac{dz}{dt} = -0.2 \, z \quad \text{with} \quad z(3) = -18
\]

Thus,

\[
z(t) = -18 \, e^{-0.2(t-3)} = y(t) - 25
\]
Example of Linear Model

The substituted model is

$$\frac{dz}{dt} = -0.2 \, z \quad \text{with} \quad z(3) = -18$$

Thus,

$$z(t) = -18 \, e^{-0.2(t-3)} = y(t) - 25$$

The solution is

$$y(t) = 25 - 18 \, e^{-0.2(t-3)}$$
Pollution in a Lake: Introduction
Pollution in a Lake: Introduction

One of the most urgent problems in modern society is how to reduce the pollution and toxicity of our water sources.
Pollution in a Lake: Introduction

One of the most urgent problems in modern society is how to reduce the pollution and toxicity of our water sources. These are very complex issues that require a multidisciplinary approach and are often politically very intractable because of the key role that water plays in human society and the many competing interests.
Pollution in a Lake: Introduction

- One of the most urgent problems in modern society is how to reduce the pollution and toxicity of our water sources.
- These are very complex issues that require a multidisciplinary approach and are often politically very intractable because of the key role that water plays in human society and the many competing interests.
- Here we examine a very simplistic model for pollution of a lake.
Pollution in a Lake: Introduction

- One of the most urgent problems in modern society is how to reduce the pollution and toxicity of our water sources.
- These are very complex issues that require a multidisciplinary approach and are often politically very intractable because of the key role that water plays in human society and the many competing interests.
- Here we examine a very simplistic model for pollution of a lake.
- The model illustrates some basic elements from which more complicated models can be built and analyzed.
Pollution in a Lake

**Pollution in a Lake:** Problem set up
Pollution in a Lake: Problem set up

Consider the scenario of a new pesticide that is applied to fields upstream from a clean lake with volume $V$. 
Pollution in a Lake: Problem set up

- Consider the scenario of a new pesticide that is applied to fields upstream from a clean lake with volume $V$.
- Assume that a river receives a constant amount of this new pesticide into its water, and that it flows into the lake at a constant rate, $f$. 

Joseph M. Mahaffy, mahaffy@math.sdsu.edu
Pollution in a Lake: Problem set up

- Consider the scenario of a new pesticide that is applied to fields upstream from a clean lake with volume $V$
- Assume that a river receives a constant amount of this new pesticide into its water, and that it flows into the lake at a constant rate, $f$
- This assumption implies that the river has a constant concentration of the new pesticide, $p$
Pollution in a Lake

Pollution in a Lake: Problem set up

- Consider the scenario of a new pesticide that is applied to fields upstream from a clean lake with volume $V$.
- Assume that a river receives a constant amount of this new pesticide into its water, and that it flows into the lake at a constant rate, $f$.
- This assumption implies that the river has a constant concentration of the new pesticide, $p$.
- Assume that the lake is well-mixed and maintains a constant volume by having a river exiting the lake with the same flow rate, $f$, of the inflowing river.
Diagram for Lake Problem Design a model using a linear first order differential equation for the concentration of the pesticide in the lake, $c(t)$

$f$: flow rate

$p$: pollutant

$V$: Volume

$c(t)$: concentration of pollutant in the lake
Pollution in a Lake

Differential Equation for Pollution in a Lake
Pollution in a Lake

**Differential Equation for Pollution in a Lake**

- Set up a differential equation that describes the mass balance of the pollutant
Pollution in a Lake

Differential Equation for Pollution in a Lake

- Set up a differential equation that describes the mass balance of the pollutant
- The change in amount of pollutant = Amount entering - Amount leaving
Differential Equation for Pollution in a Lake

- Set up a differential equation that describes the mass balance of the pollutant

- The change in amount of pollutant = Amount entering - Amount leaving

- The amount entering is simply the concentration of the pollutant, $p$, in the river times the flow rate of the river, $f$
Pollution in a Lake

**Differential Equation for Pollution in a Lake**

- Set up a differential equation that describes the mass balance of the pollutant

- **The change in amount of pollutant = Amount entering - Amount leaving**

- The amount entering is simply the concentration of the pollutant, \( p \), in the river times the flow rate of the river, \( f \)

- The amount leaving has the same flow rate, \( f \)
Pollution in a Lake

Differential Equation for Pollution in a Lake

- Set up a differential equation that describes the mass balance of the pollutant
- The change in amount of pollutant = Amount entering - Amount leaving
- The amount entering is simply the concentration of the pollutant, $p$, in the river times the flow rate of the river, $f$
- The amount leaving has the same flow rate, $f$
- Since the lake is assumed to be well-mixed, the concentration in the outflowing river will be equal to the concentration of the pollutant in the lake, $c(t)$
Pollution in a Lake

Differential Equation for Pollution in a Lake

- Set up a differential equation that describes the mass balance of the pollutant
- **The change in amount of pollutant = Amount entering - Amount leaving**
- The amount entering is simply the concentration of the pollutant, $p$, in the river times the flow rate of the river, $f$
- The amount leaving has the same flow rate, $f$
- Since the lake is assumed to be well-mixed, the concentration in the outflowing river will be equal to the concentration of the pollutant in the lake, $c(t)$
- The product $f c(t)$ gives the amount of pollutant leaving the lake per unit time
Introduction
Blood Pressure
Radioactive Decay
Solution of Linear Growth and Decay Models
Newton’s Law of Cooling
Solution of General Linear Model
Pollution in a Lake

Pollution in a Lake

Differential Equations for Amount and Concentration of Pollutant
Differential Equations for Amount and Concentration of Pollutant

- The change in amount of pollutant satisfies the model

\[
\frac{da(t)}{dt} = fp - fc(t)
\]
Differential Equations for Amount and Concentration of Pollutant

- The change in amount of pollutant satisfies the model
  \[ \frac{da(t)}{dt} = f_p - f_c(t) \]

- Since the lake maintains a constant volume \( V \), then \( c(t) = \frac{a(t)}{V} \), which also implies that \( c'(t) = \frac{a'(t)}{V} \)
Pollution in a Lake

Differential Equations for Amount and Concentration of Pollutant

- The change in amount of pollutant satisfies the model
  \[
  \frac{da(t)}{dt} = fp - fc(t)
  \]

- Since the lake maintains a constant volume \( V \), then
  \( c(t) = a(t)/V \), which also implies that \( c'(t) = a'(t)/V \)

- Dividing the above differential equation by the volume \( V \),
  \[
  \frac{dc(t)}{dt} = \frac{f}{V} (p - c(t))
  \]
Pollution in a Lake

Differential Equations for Amount and Concentration of Pollutant

- The change in amount of pollutant satisfies the model
  \[
  \frac{da(t)}{dt} = fp - fc(t)
  \]

- Since the lake maintains a constant volume \( V \), then \( c(t) = a(t)/V \), which also implies that \( c'(t) = a'(t)/V \)

- Dividing the above differential equation by the volume \( V \),
  \[
  \frac{dc(t)}{dt} = \frac{f}{V}(p - c(t))
  \]

- If the lake is initially clean, then \( c(0) = 0 \)
Pollution in a Lake

**Solution of the Differential Equation:** Rewrite the differential equation for the concentration of pollutant as

\[
\frac{dc(t)}{dt} = -\frac{f}{V} (c(t) - p) \quad \text{with} \quad c(0) = 0
\]
Pollution in a Lake

Soluton of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

$$\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0$$

This DE should remind you of Newton’s Law of Cooling with $f/V$ acting like $k$ and $p$ acting like $T_e$
Soluton of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

\[ \frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0 \]

- This DE should remind you of Newton’s Law of Cooling with \( f/V \) acting like \( k \) and \( p \) acting like \( T_e \)
- Make the substitution, \( z(t) = c(t) - p \), so \( z'(t) = c'(t) \)
Solution of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

\[
\frac{dc(t)}{dt} = -\frac{f}{V} (c(t) - p) \quad \text{with} \quad c(0) = 0
\]

- This DE should remind you of Newton’s Law of Cooling with \( f/V \) acting like \( k \) and \( p \) acting like \( T_e \)
- Make the substitution, \( z(t) = c(t) - p \), so \( z'(t) = c'(t) \)
- The initial condition becomes \( z(0) = c(0) - p = -p \)
Pollution in a Lake

Solution of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

\[
\frac{dc(t)}{dt} = -\frac{f}{V} (c(t) - p) \quad \text{with} \quad c(0) = 0
\]

- This DE should remind you of Newton’s Law of Cooling with \( f/V \) acting like \( k \) and \( p \) acting like \( T_e \).
- Make the substitution, \( z(t) = c(t) - p \), so \( z'(t) = c'(t) \).
- The initial condition becomes \( z(0) = c(0) - p = -p \).
- The initial value problem in \( z(t) \) becomes,

\[
\frac{dz(t)}{dt} = -\frac{f}{V} z(t), \quad \text{with} \quad z(0) = -p
\]
Solution of the Differential Equation (cont): Since
\[ \frac{dz(t)}{dt} = -\frac{f}{V}z(t), \quad \text{with} \quad z(0) = -p \]
Solution of the Differential Equation (cont): Since

\[
\frac{dz(t)}{dt} = -\frac{f}{V} z(t), \quad \text{with} \quad z(0) = -p
\]

- The solution to this problem is

\[
z(t) = -p e^{-\frac{ft}{V}} = c(t) - p
\]
Pollution in a Lake

Solution of the Differential Equation (cont): Since

$$\frac{dz(t)}{dt} = -\frac{f}{V} z(t), \text{ with } \ z(0) = -p$$

• The solution to this problem is

$$z(t) = -p e^{-\frac{ft}{V}} = c(t) - p$$

•

$$c(t) = p \left(1 - e^{-\frac{ft}{V}}\right)$$
Pollution in a Lake

Solution of the Differential Equation (cont): Since

$$\frac{dz(t)}{dt} = -\frac{f}{V} z(t), \quad \text{with} \quad z(0) = -p$$

- The solution to this problem is

$$z(t) = -p e^{-\frac{ft}{V}} = c(t) - p$$

- $$c(t) = p \left(1 - e^{-\frac{ft}{V}}\right)$$

- The exponential decay in this solution shows

$$\lim_{t \to \infty} c(t) = p$$
Pollution in a Lake

Solution of the Differential Equation (cont): Since

\[ \frac{dz(t)}{dt} = -\frac{f}{V} z(t), \quad \text{with} \quad z(0) = -p \]

- The solution to this problem is

\[ z(t) = -p e^{-\frac{ft}{V}} = c(t) - p \]

- The exponential decay in this solution shows

\[ \lim_{t \to \infty} c(t) = p \]

- This is exactly what you would expect, as the entering river has a concentration of \( p \)
Example: Pollution in a Lake Part 1
Example: Pollution in a Lake Part 1

- Suppose that you begin with a 10,000 m$^3$ clean lake
Example: Pollution in a Lake Part 1

- Suppose that you begin with a 10,000 m$^3$ clean lake
- Assume the river entering has a flow of 100 m$^3$/day and the concentration of some pesticide in the river is measured to have a concentration of 5 ppm (parts per million)
Example: Pollution in a Lake Part 1

- Suppose that you begin with a 10,000 m$^3$ clean lake
- Assume the river entering has a flow of 100 m$^3$/day and the concentration of some pesticide in the river is measured to have a concentration of 5 ppm (parts per million)
- Form the differential equation describing the concentration of pollutant in the lake at any time $t$ and solve it
Example: Pollution in a Lake Part 1

- Suppose that you begin with a 10,000 m$^3$ clean lake
- Assume the river entering has a flow of 100 m$^3$/day and the concentration of some pesticide in the river is measured to have a concentration of 5 ppm (parts per million)
- Form the differential equation describing the concentration of pollutant in the lake at any time $t$ and solve it
- Find out how long it takes for this lake to have a concentration of 2 ppm
Example: Pollution in a Lake

Solution: This example follows the model derived above, so the differential equation for the concentration of pollutant is

\[
\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0
\]
Example: Pollution in a Lake

Solution: This example follows the model derived above, so the differential equation for the concentration of pollutant is

\[
\frac{dc(t)}{dt} = -\frac{f}{V} (c(t) - p) \quad \text{with} \quad c(0) = 0
\]

Since \( V = 10,000 \), \( f = 100 \), and \( p = 5 \),

\[
\frac{dc(t)}{dt} = -\frac{100}{10000} (c(t) - 5) \quad \text{with} \quad c(0) = 0
\]
Example: Pollution in a Lake

**Solution:** This example follows the model derived above, so the differential equation for the concentration of pollutant is

\[
\frac{dc(t)}{dt} = -\frac{f}{V} (c(t) - p) \quad \text{with} \quad c(0) = 0
\]

- Since \( V = 10,000 \), \( f = 100 \), and \( p = 5 \),

\[
\frac{dc(t)}{dt} = -\frac{100}{10000} (c(t) - 5) \quad \text{with} \quad c(0) = 0
\]

- Let \( z(t) = c(t) - 5 \), then the differential equation becomes,

\[
\frac{dz}{dt} = -0.01z(t), \quad \text{with} \quad z(0) = -5
\]
Solution: This example follows the model derived above, so the differential equation for the concentration of pollutant is

\[
\frac{dc(t)}{dt} = -\frac{f}{V} (c(t) - p) \quad \text{with} \quad c(0) = 0
\]

- Since \( V = 10,000 \), \( f = 100 \), and \( p = 5 \),

\[
\frac{dc(t)}{dt} = -\frac{100}{10000} (c(t) - 5) \quad \text{with} \quad c(0) = 0
\]

- Let \( z(t) = c(t) - 5 \), then the differential equation becomes,

\[
\frac{dz}{dt} = -0.01z(t), \quad \text{with} \quad z(0) = -5
\]

- This has a solution

\[
z(t) = -5e^{-0.01t} = c(t) - 5
\]
Example: Pollution in a Lake

Solution (cont): The concentration of pollutant in the lake is

\[ c(t) = 5 \left( 1 - e^{-0.01t} \right) \]
Example: Pollution in a Lake

Solution (cont): The concentration of pollutant in the lake is

\[ c(t) = 5 \left( 1 - e^{-0.01t} \right) \]

To find how long it takes for the concentration to reach 2 ppm, solve the equation

\[ 2 = 5 - 5e^{-0.01t} \]
Example: Pollution in a Lake

Solution (cont): The concentration of pollutant in the lake is

\[ c(t) = 5 \left( 1 - e^{-0.01t} \right) \]

- To find how long it takes for the concentration to reach 2 ppm, solve the equation

\[ 2 = 5 - 5e^{-0.01t} \]

- Thus,

\[ e^{-0.01t} = \frac{3}{5} \quad \text{or} \quad e^{0.01t} = \frac{5}{3} \]
**Example: Pollution in a Lake**

**Solution (cont):** The concentration of pollutant in the lake is

\[ c(t) = 5 \left( 1 - e^{-0.01t} \right) \]

- To find how long it takes for the concentration to reach 2 ppm, solve the equation

\[ 2 = 5 - 5e^{-0.01t} \]

- Thus,

\[ e^{-0.01t} = \frac{3}{5} \quad \text{or} \quad e^{0.01t} = \frac{5}{3} \]

- Solving this for \( t \), we obtain

\[ t = 100 \ln \left( \frac{5}{3} \right) = 51.1 \text{ days} \]
Example: Pollution in a Lake Part 2
Example: Pollution in a Lake Part 2

Suppose that when the concentration reaches 4 ppm, the pesticide is banned.
Example: Pollution in a Lake Part 2

- Suppose that when the concentration reaches 4 ppm, the pesticide is banned
- For simplicity, assume that the concentration of pesticide drops immediately to zero in the river
Example: Pollution in a Lake Part 2

- Suppose that when the concentration reaches 4 ppm, the pesticide is banned.
- For simplicity, assume that the concentration of pesticide drops immediately to zero in the river.
- Assume that the pesticide is not degraded or lost by any means other than dilution.
Example: Pollution in a Lake Part 2

- Suppose that when the concentration reaches 4 ppm, the pesticide is banned.
- For simplicity, assume that the concentration of pesticide drops immediately to zero in the river.
- Assume that the pesticide is not degraded or lost by any means other than dilution.
- Find how long until the concentration reaches 1 ppm.
Example: Pollution in a Lake

Solution: The new initial value problem becomes

\[ \frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4 \]
Example: Pollution in a Lake

**Solution:** The new initial value problem becomes

\[
\frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4
\]

This problem is in the form of a radioactive decay problem.
Example: Pollution in a Lake

**Solution:** The new initial value problem becomes

\[ \frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4 \]

- This problem is in the form of a radioactive decay problem
- This has the solution

\[ c(t) = 4e^{-0.01t} \]
Example: Pollution in a Lake

**Solution:** The new initial value problem becomes

\[
\frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4
\]

- This problem is in the form of a radioactive decay problem
- This has the solution

\[
c(t) = 4e^{-0.01t}
\]

- To find how long it takes for the concentration to return to 1 ppm, solve the equation

\[
1 = 4e^{-0.01t} \quad \text{or} \quad e^{0.01t} = 4
\]
**Example: Pollution in a Lake**

**Solution:** The new initial value problem becomes

\[
\frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4
\]

- This problem is in the form of a radioactive decay problem
- This has the solution
  \[c(t) = 4e^{-0.01t}\]

- To find how long it takes for the concentration to return to 1 ppm, solve the equation
  \[1 = 4e^{-0.01t} \quad \text{or} \quad e^{0.01t} = 4\]

- Solving this for \(t\)
  \[t = 100 \ln(4) = 138.6 \text{ days}\]
Pollution in a Lake: Complications

Pollution in a Lake: Complications The above discussion for pollution in a lake fails to account for many significant complications.
Pollution in a Lake: Complications

The above discussion for pollution in a lake fails to account for many significant complications.

There are considerations of degradation of the pesticide, stratification in the lake, and uptake and reentering of the pesticide through interaction with the organisms living in the lake.
Pollution in a Lake: Complications

The above discussion for pollution in a lake fails to account for many significant complications:

- There are considerations of degradation of the pesticide, stratification in the lake, and uptake and reentering of the pesticide through interaction with the organisms living in the lake.

- The river will vary in its flow rate, and the leaching of the pesticide into river is highly dependent on rainfall, ground water movement, and rate of pesticide application.
Pollution in a Lake: Complications

The above discussion for pollution in a lake fails to account for many significant complications:

- There are considerations of degradation of the pesticide, stratification in the lake, and uptake and reentering of the pesticide through interaction with the organisms living in the lake.

- The river will vary in its flow rate, and the leaching of the pesticide into river is highly dependent on rainfall, groundwater movement, and rate of pesticide application.

- Obviously, there are many other complications that would increase the difficulty of analyzing this model.
Pollution in a Lake: Complications

The above discussion for pollution in a lake fails to account for many significant complications:

- There are considerations of degradation of the pesticide, stratification in the lake, and uptake and reentering of the pesticide through interaction with the organisms living in the lake.

- The river will vary in its flow rate, and the leaching of the pesticide into river is highly dependent on rainfall, ground water movement, and rate of pesticide application.

- Obviously, there are many other complications that would increase the difficulty of analyzing this model.

- The next section shows numerical methods to handle more complicated models.
Example: Lake Pollution with Evaporation
Example: Lake Pollution with Evaporation

Suppose that a new industry starts up river from a lake at $t = 0$ days, and this industry starts dumping a toxic pollutant, $P(t)$, into the river at a rate of 7 g/day, which flows directly into the lake.
Example: Lake Pollution with Evaporation

Suppose that a new industry starts up river from a lake at \( t = 0 \) days, and this industry starts dumping a toxic pollutant, \( P(t) \), into the river at a rate of 7 g/day, which flows directly into the lake.

The flow of the river is 1000 m\(^3\)/day, which goes into the lake that maintains a constant volume of 400,000 m\(^3\).
Example: Lake Pollution with Evaporation

Suppose that a new industry starts up river from a lake at \( t = 0 \) days, and this industry starts dumping a toxic pollutant, \( P(t) \), into the river at a rate of 7 g/day, which flows directly into the lake.

The flow of the river is 1000 m\(^3\)/day, which goes into the lake that maintains a constant volume of 400,000 m\(^3\).

The lake is situated in a hot area and loses 50 m\(^3\)/day of water to evaporation (pure water with no pollutant), while the remainder of the water exits at a rate of 950 m\(^3\)/day through a river.
Example: Lake Pollution with Evaporation

Suppose that a new industry starts up river from a lake at \( t = 0 \) days, and this industry starts dumping a toxic pollutant, \( P(t) \), into the river at a rate of 7 g/day, which flows directly into the lake.

- The flow of the river is 1000 m\(^3\)/day, which goes into the lake that maintains a constant volume of 400,000 m\(^3\).
- The lake is situated in a hot area and loses 50 m\(^3\)/day of water to evaporation (pure water with no pollutant), while the remainder of the water exits at a rate of 950 m\(^3\)/day through a river.
- Assume that all quantities are well-mixed and that there are no time delays for the pollutant reaching the lake from the river.
Example: Lake Pollution with Evaporation (cont) Part a
Example: Lake Pollution with Evaporation (cont) Part a

- Write a differential equation that describes the concentration, \( c(t) \), of the pollutant in the lake, using units of mg/m\(^3\)
Example: Lake Pollution with Evaporation (cont) Part a

- Write a differential equation that describes the concentration, \( c(t) \), of the pollutant in the lake, using units of \( \text{mg/m}^3 \)
- Solve the differential equation
Example: Lake Pollution with Evaporation (cont) Part a

- Write a differential equation that describes the concentration, $c(t)$, of the pollutant in the lake, using units of mg/m$^3$.
- Solve the differential equation.
- If a concentration of only 2 mg/m$^3$ is toxic to the fish population, then find how long until this level is reached.
Example: Lake Pollution with Evaporation (cont) Part a

- Write a differential equation that describes the concentration, \( c(t) \), of the pollutant in the lake, using units of \( \text{mg/m}^3 \)
- Solve the differential equation
- If a concentration of only 2 \( \text{mg/m}^3 \) is toxic to the fish population, then find how long until this level is reached
- If unchecked by regulations, then find what the eventual concentration of the pollutant is in the lake, assuming constant output by the new industry
Example: Lake Pollution with Evaporation

Solution: Let $P(t)$ be the amount of pollutant.
The change in amount of pollutant = Amount entering - Amount leaving.
Example: Lake Pollution with Evaporation

Solution: Let $P(t)$ be the amount of pollutant.

The change in amount of pollutant =
Amount entering - Amount leaving

- The change in amount is $\frac{dP}{dt}$
Example: Lake Pollution with Evaporation

**Solution:** Let $P(t)$ be the amount of pollutant.

The change in amount of pollutant =  
Amount entering - Amount leaving

- The change in amount is $\frac{dP}{dt}$
- The concentration is given by $c(t) = \frac{P(t)}{V}$ and $c'(t) = \frac{P'(t)}{V}$
Example: Lake Pollution with Evaporation

Solution: Let $P(t)$ be the amount of pollutant.

The change in amount of pollutant =

Amount entering - Amount leaving

- The change in amount is $\frac{dP}{dt}$
- The concentration is given by $c(t) = \frac{P(t)}{V}$ and $c'(t) = \frac{P'(t)}{V}$
- The amount entering is the constant rate of pollutant dumped into the river, which is given by $k = 7000 \text{ mg/day}$
Example: Lake Pollution with Evaporation

**Solution:** Let $P(t)$ be the amount of pollutant

The change in amount of pollutant =

Amount entering - Amount leaving

- The **change in amount** is $\frac{dP}{dt}$
- The concentration is given by $c(t) = \frac{P(t)}{V}$ and $c'(t) = \frac{P'(t)}{V}$
- The **amount entering** is the constant rate of pollutant dumped into the river, which is given by $k = 7000$ mg/day
- The **amount leaving** is given by the concentration of the pollutant in the lake, $c(t)$ (in mg/m$^3$), times the flow of water out of the lake, $f = 950$ m$^3$/day
Example: Lake Pollution with Evaporation

Solution (cont): The conservation of amount of pollutant is given by the equation:

$$\frac{dP}{dt} = k - f c(t) = 7000 - 950c(t)$$
Example: Lake Pollution with Evaporation

Solution (cont): The conservation of amount of pollutant is given by the equation:

\[ \frac{dP}{dt} = k - f c(t) = 7000 - 950 c(t) \]

- Evaporation concentrates the pollutant by allowing water to leave without the pollutant
Example: Lake Pollution with Evaporation

Solution (cont): The conservation of amount of pollutant is given by the equation:

\[
\frac{dP}{dt} = k - f c(t) = 7000 - 950 c(t)
\]

- Evaporation concentrates the pollutant by allowing water to leave without the pollutant.
- Divide the equation above by the volume, \( V = 400,000 \text{ m}^3 \)

\[
\left( \frac{1}{V} \right) \frac{dP(t)}{dt} = \frac{k}{V} - \frac{f}{V} c(t) = \frac{7}{400} - \frac{950}{400000} c(t)
\]
Example: Lake Pollution with Evaporation

Solution (cont): The conservation of amount of pollutant is given by the equation:

\[
\frac{dP}{dt} = k - f c(t) = 7000 - 950 c(t)
\]

- Evaporation concentrates the pollutant by allowing water to leave without the pollutant
- Divide the equation above by the volume, \( V = 400,000 \) m³

\[
\left( \frac{1}{V} \right) \frac{dP(t)}{dt} = \frac{k}{V} - \frac{f}{V} c(t) = \frac{7}{400} - \frac{950}{400000} c(t)
\]

- The concentration equation is

\[
\frac{dc}{dt} = \frac{7}{400} - \frac{950}{400000} c(t) = -\frac{f}{V} \left( c(t) - \frac{k}{f} \right)
\]
Example: Lake Pollution with Evaporation

Solution (cont): The concentration equation is

\[
\frac{dc}{dt} = -\frac{95}{40000} \left( c(t) - \frac{700}{95} \right)
\]
Solution (cont): The concentration equation is

\[
\frac{dc}{dt} = -\frac{95}{40000} \left( c(t) - \frac{700}{95} \right)
\]

Make the change of variables, \( z(t) = c(t) - \frac{700}{95} \), with \( z(0) = -\frac{700}{95} \)
Solution (cont): The concentration equation is

\[ \frac{dc}{dt} = -\frac{95}{40000} (c(t) - \frac{700}{95}) \]

- Make the change of variables, \( z(t) = c(t) - \frac{700}{95} \), with \( z(0) = -\frac{700}{95} \)
- The differential equation is

\[ \frac{dz}{dt} = -\frac{95}{40000} z(t) \quad \text{with} \quad z(0) = -\frac{700}{95} \]
Example: Lake Pollution with Evaporation

Solution (cont): The concentration equation is

\[
\frac{dc}{dt} = -\frac{95}{40000} \left( c(t) - \frac{700}{95} \right)
\]

- Make the change of variables, \( z(t) = c(t) - \frac{700}{95} \), with \( z(0) = -\frac{700}{95} \)
- The differential equation is

\[
\frac{dz}{dt} = -\frac{95}{40000} z(t) \quad \text{with} \quad z(0) = -\frac{700}{95}
\]

- The solution is

\[
z(t) = -\frac{700}{95} e^{-95t/40000} = c(t) - \frac{700}{95}
\]
Example: Lake Pollution with Evaporation

Solution (cont): The concentration equation is

\[ c(t) = \frac{700}{95} \left( 1 - e^{-95t/40000} \right) \approx 7.368 \left( 1 - e^{-0.002375t} \right) \]
Solution (cont): The concentration equation is
\[ c(t) = \frac{700}{95} \left( 1 - e^{-95t/40000} \right) \approx 7.368 \left( 1 - e^{-0.002375t} \right) \]

- If a concentration of 2 mg/m³ is toxic to the fish population, then find when \( c(t) = 2 \) mg/m³
Example: Lake Pollution with Evaporation

Solution (cont): The concentration equation is

\[ c(t) = \frac{700}{95} \left(1 - e^{-\frac{95t}{40000}}\right) \approx 7.368 \left(1 - e^{-0.002375t}\right) \]

- If a concentration of 2 mg/m³ is toxic to the fish population, then find when \( c(t) = 2 \text{ mg/m}^3 \)
- Solve

\[ 2 = 7.368 \left(1 - e^{-0.002375t}\right) \quad \text{or} \quad e^{0.002375t} \approx 1.3726 \]
Example: Lake Pollution with Evaporation

Solution (cont): The concentration equation is

\[ c(t) = \frac{700}{95} \left(1 - e^{-95t/40000}\right) \approx 7.368 \left(1 - e^{-0.002375t}\right) \]

- If a concentration of 2 mg/m³ is toxic to the fish population, then find when \( c(t) = 2 \) mg/m³
- Solve

\[ 2 = 7.368 \left(1 - e^{-0.002375t}\right) \quad \text{or} \quad e^{0.002375t} \approx 1.3726 \]

- Thus, \( t = \frac{\ln(1.3726)}{0.002375} \approx 133.3 \) days
Solution (cont): The concentration equation is
\[ c(t) = \frac{700}{95} \left( 1 - e^{-\frac{95t}{40000}} \right) \approx 7.368 \left( 1 - e^{-0.002375t} \right) \]

- If a concentration of 2 mg/m\(^3\) is toxic to the fish population, then find when \( c(t) = 2 \text{ mg/m}^3 \)
- Solve
\[
2 = 7.368 \left( 1 - e^{-0.002375t} \right) \quad \text{or} \quad e^{0.002375t} \approx 1.3726
\]
- Thus, \( t = \frac{\ln(1.3726)}{0.002375} \approx 133.3 \text{ days} \)
- The limiting concentration is
\[
\lim_{t \to \infty} c(t) = \frac{700}{95} \approx 7.368
\]
Example: Lake Pollution with Evaporation (cont) Part b
Example: Lake Pollution with Evaporation (cont) Part b

Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time $t = 0$ days.
Example: Lake Pollution with Evaporation (cont) Part b

- Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time $t = 0$ days.

- Write a new differential equation describing the situation following the shutdown of the industry and solve this equation.
Example: Lake Pollution with Evaporation (cont) Part b

- Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time $t = 0$ days.

- Write a new differential equation describing the situation following the shutdown of the industry and solve this equation.

- Calculate how long it takes for the lake to return to a level that allows fish to survive.
Solution: Now $k = 0$, so the initial value problem becomes

$$\frac{dc}{dt} = -\frac{95}{40000} c(t) = -0.002375 c(t) \quad \text{with} \quad c(0) = \frac{700}{95}$$
Example: Lake Pollution with Evaporation

**Solution:** Now $k = 0$, so the initial value problem becomes

$$\frac{dc}{dt} = -\frac{95}{40000} c(t) = -0.002375 c(t) \quad \text{with} \quad c(0) = \frac{700}{95}$$

- This has the solution

$$c(t) = \frac{700}{95} e^{-0.002375t} \approx 7.368 e^{-0.002375t}$$
Example: Lake Pollution with Evaporation

Solution: Now $k = 0$, so the initial value problem becomes

$$\frac{dc}{dt} = -\frac{95}{40000} c(t) = -0.002375 c(t) \quad \text{with} \quad c(0) = \frac{700}{95}$$

• This has the solution

$$c(t) = \frac{700}{95} e^{-0.002375 t} \approx 7.368 e^{-0.002375 t}$$

• The concentration is reduced to 2 mg/m$^3$ when

$$2 = 7.368 e^{-0.002375 t} \quad \text{or} \quad e^{0.002375 t} = 3.684$$
Example: Lake Pollution with Evaporation

**Solution:** Now $k = 0$, so the initial value problem becomes

$$\frac{dc}{dt} = -\frac{95}{40000}c(t) = -0.002375c(t) \quad \text{with} \quad c(0) = \frac{700}{95}$$

- This has the solution
  $$c(t) = \frac{700}{95} e^{-0.002375t} \approx 7.368 e^{-0.002375t}$$

- The concentration is reduced to 2 mg/m$^3$ when
  $$2 = 7.368 e^{-0.002375t} \quad \text{or} \quad e^{0.002375t} = 3.684$$

- The lake is sufficiently clean for fish when
  $$t = \frac{\ln(3.684)}{0.002375} \approx 549 \text{ days}$$