1. a. In class we developed the system of differential equations, which describes the motion of the pendulum attached to a spring, and these equations are given by the equations:

$$\begin{aligned} L\frac{d^2\theta}{dt^2} + 2\frac{dL}{dt}\frac{d\theta}{dt} &= -g\sin(\theta), \\ \frac{d^2L}{dt^2} - L\left(\frac{d\theta}{dt}\right)^2 &= g\cos(\theta) - \frac{k}{m}(L - L_0), \end{aligned}$$

where L is the total length from the pivot to the mass m, L_0 is the unstretched length of the spring, θ is the angle from the vertical of the pendulum, and k is the spring constant for the system. (Friction is ignored in this model.) Let $y_1(t) = L(t)$, $y_2(t) = \dot{L}(t)$, $y_3(t) = \theta(t)$, and $y_4(t) = \dot{\theta}(t)$ and form a system of first order nonlinear differential equations.

b. Find all equilibria for this system.

c. Determine the linearized system about all equilibria.

d. Find the characteristic equations and determine eigenvalues.

e. Find the periods of motion from the linear system (with imaginary eigenvalues) in the L direction and in the θ direction.

2. a. Consider the spring-pendulum that you observed in class (films available on the web). The lecture notes have the key parameters that you need for your simulation. In particular, the mass of the ball is $m = 233$ g, (the spring itself weighs 155 g, but should be ignored in this part of the model), $\omega^2 = k/m$ when air resistance is ignored, $L_0 = 19.75$ cm, and the stretched length with the ball attached is 53.5 cm. Use $g = 980$ cm/sec^2 and find the value of k and ω.

b. If $L(t)$ is the length from the pivot to the center of the steel ball and $\theta(t)$ is the angle the spring-pendulum makes with the vertical axis, then write the complete first order system of nonlinear differential equations describing the motion of the spring-pendulum problem with these parameter values.

c. Simulate the motion for this nonlinear system for 10 sec, given the initial conditions, $L(0) = 60$ cm and $\theta(0) = \pi/6$. Also, assume that the mass starts from rest, so the velocity is initially zero. Provide a plot of L vs t, θ vs t, and a polar plot of L vs θ.

d. Briefly discuss at least two additional factors that could be added to the model to improve the simulation toward matching real data.