1. In class we examined basic models for fishing with constant limits and proportional harvesting. Below we outline a model for the population of fish with a Holling's Type II term for the harvesting of fish. Suppose that a population of fish, $F(t)$ (in thousands), is given by the following model

$$\frac{dF}{dt} = 0.2\,F\left(1 - \frac{F}{100}\right) - \frac{hF}{1 + 0.02F},$$

where h is the harvesting term from fishing.

a. Give a modeling description of each term in the equation above. Compare this model to the two models studied in lecture with constant limits and proportional fishing.

b. Assume there is no fishing ($h = 0$). Find all equilibria for this model. Sketch a graph of the right hand side of the model, then draw the phase portrait. Determine the stability of all equilibria.

c. Let $h = 0.05$, then find all equilibria for this model. Sketch a graph of the right hand side of the model, then draw the phase portrait. Determine the stability of all equilibria.

d. What level of fishing (value of h) results in the fish going extinct? What type of bifurcation occurs at this value of h?

2. Use the analysis of the Lotka-Volterra or Predator-Prey model to explain why the application of DDT to control scale insects in the citrus industry led to excessive application of the pesticide, which in turn resulted in many environment problems highlighted in the classic book that really started the environmental movement, Rachel Carlson's *Silent Spring* (referring to the devastation of the bird populations). The citrus groves were rapidly invaded by scale insects, which caused tremendous destruction until lady bugs, the scale insect's natural predator, were imported as a control. Give at least two reasons why use of pesticides result in escalating use of pesticides that further put farmers in debt and only enrich the chemical industry. Use modeling methods to show some smarter way to control an agricultural pest.

3. In this problem, you extend modeling efforts of the Lotka-Volterra/Predator-Prey model to examine two variants of the classical model (which we showed to be neutrally stable giving an unstable model).

a. An alternative predator-prey model is often used, where the model includes an intraspecies competition term for the prey species (often considered the next most important modeling term). This model is given by the system of differential equations

$$\begin{aligned}\dot{H} &= a_1H - a_2HL - a_3H^2,\\ \dot{L} &= -b1L + b_2HL.\end{aligned}$$

This system has an additional parameter, so how does this change the analysis of the model.

Find all equilibria for this model, then discuss the stability of these equilibria. Give the condition that makes one of the equilibria strictly positive in both the predator and prey populations and use this condition in your stability analysis. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral). Is this model structurally stable?

b. Another predator-prey model considers the fact that the prey population could satiate the predator, so a Holling's Type II term for predation is used. This model is given by the system of differential equations

$$\begin{aligned}\dot{H} &= a_1 H - \frac{a_2 H L}{1 + k_1 H}, \\ \dot{L} &= -b1 L + \frac{b_2 H L}{1 + k_1 H}.\end{aligned}$$

This system has an additional parameter, so how does this change the analysis of the model.

Find all equilibria for this model, then discuss the stability of these equilibria. Give the condition that makes one of the equilibria strictly positive in both the predator and prey populations and use this condition in your stability analysis. Characterize each of the equilibria (*e.g.*, stable node, saddle node, unstable spiral). Is this model structurally stable?

4. In lecture we fit 20 years of data from the Hudson Bay Company. Below we extend the model and consider the two models above to fit 30 years of data from the Hudson Bay Company (extending 10 more years). Below is a table of the 30 years of pelt data from the Hudson Bay Company.

Year	Hare ($\times$1000)	Lynx ($\times$1000)	Year	Hare ($\times$1000)	Lynx ($\times$1000)
1900	30	4	1916	11.2	29.7
1901	47.2	6.1	1917	7.6	15.8
1902	70.2	9.8	1918	14.6	9.7
1903	77.4	35.2	1919	16.2	10.1
1904	36.3	59.4	1920	24.7	8.6
1905	20.6	41.7	1921	33	18.8
1906	18.1	19	1922	37.1	25.6
1907	21.4	13	1923	41.2	34.2
1908	22	8.3	1924	35.7	41.9
1909	25.4	9.1	1925	14.9	47
1910	27.1	7.4	1926	4.5	53
1911	40.3	8	1927	0.9	42.7
1912	57	12.3	1928	0.9	31.6
1913	76.6	19.5	1929	1.8	20.5
1914	52.3	45.7	1930	2.7	10.3
1915	19.5	51.1			

a. For this part of the problem, you repeat the work done in the lecture notes with the Lotka-Volterra model for the lynx and hare study

$$\begin{aligned}\dot{H} &= a_1 H - a_2 H L, \\ \dot{L} &= -b1 L + b_2 H L.\end{aligned}$$

Begin with the parameters values from the best fit for the first 20 years of data, then use MatLab's fminsearch to find the least squares best fit to the parameters and initial conditions, $H(0)$, $L(0)$, a_1, a_2, b_1, and b_2. (Let $t = 0$ be 1900.) Give the best sum of square errors and the values for the best parameters and initial conditions. Create a time series graph showing the data (both lynx and hares), the simulation using the parameters and initial conditions from the 20 year best fit (from the notes), and the simulation with the best fitting parameters and initial conditions. Also, create a phase portrait (2D) showing the data, the simulation from the 20 year best fit, and the simulation from the 30 year best fit. (Be sure to include arrows to show the direction of the flow of the solution.) Write a brief discussion of how well your model simulates the actual data.

b. Next you take the modified model given in Problem 2 given by the equations:

$$\begin{aligned} \dot{H} &= a_1 H - a_2 HL - a_3 H^2, \\ \dot{L} &= -b1L + b_2 HL. \end{aligned}$$

You now have the additional parameter a_3 added to the model. Start with the best fitting parameters from Part a. In addition, a_3 relates to the carrying capacity of hares in the absence of lynx, so use an initial estimate on a_3 that corresponds to at least twice the maximum observed population of hares. Give the best sum of square errors and the values for the best fitting parameters and initial conditions. Create a time series graph showing the data (both lynx and hares), the simulation from Part a (with best parameters), and the simulation with this modified predator-prey model and its best fitting parameters and initial conditions. Also, create a phase portrait (2D) showing the data, the simulation from the best fitting parameters and initial conditions in Part a, and the simulation for the best fitting parameters and initial conditions for this modified model.

c. For the model that you found in Part b, find all equilibria and find the eigenvalues at those equilibria. Graph the nullclines, clearly labeling the equilibria. Show the direction field for this system of differential equations (being sure that the nullclines are clearly visible). Discuss the stability of each of the equilibria and predict what the model predicts will happen with the populations of the lynx and hares. Is this prediction reasonable? Write a brief discussion of how well this model simulates the actual data and how it compares to the model in Part a. Clearly state your reasoning.

d. Now consider the model with the Holling's Type II model given by:

$$\begin{aligned} \dot{H} &= a_1 H - \frac{a_2 HL}{1 + k_1 H}, \\ \dot{L} &= -b1L + \frac{b_2 HL}{1 + k_1 H}. \end{aligned}$$

Repeat the work you did in Part b for the model with a carrying capacity for the prey. In this case, you will want to insert k_1^2 for k_1 in the MatLab program to insure that your parameter is positive. Initially, you choose k_1 very small. Find the sum of square errors and the best fitting parameters. Repeat the graphs and discussion that you did in Part b.

e. For the model that you found in Part d, find all equilibria and find the eigenvalues at those equilibria. Graph the nullclines, clearly labeling the equilibria. Show the direction field for this system of differential equations (being sure that the nullclines are clearly visible). Discuss the stability of each of the equilibria and predict what the model predicts will happen with the populations of the lynx and hares. Is this prediction reasonable? Compare all three models (Parts a, b, and d) giving why you suspect one is superior to another. Clearly state your reasoning.