Numerical Differentiation and Integration

Composite Numerical Integration; Romberg Integration
Adaptive Quadrature / Gaussian Quadrature

Lecture Notes #8

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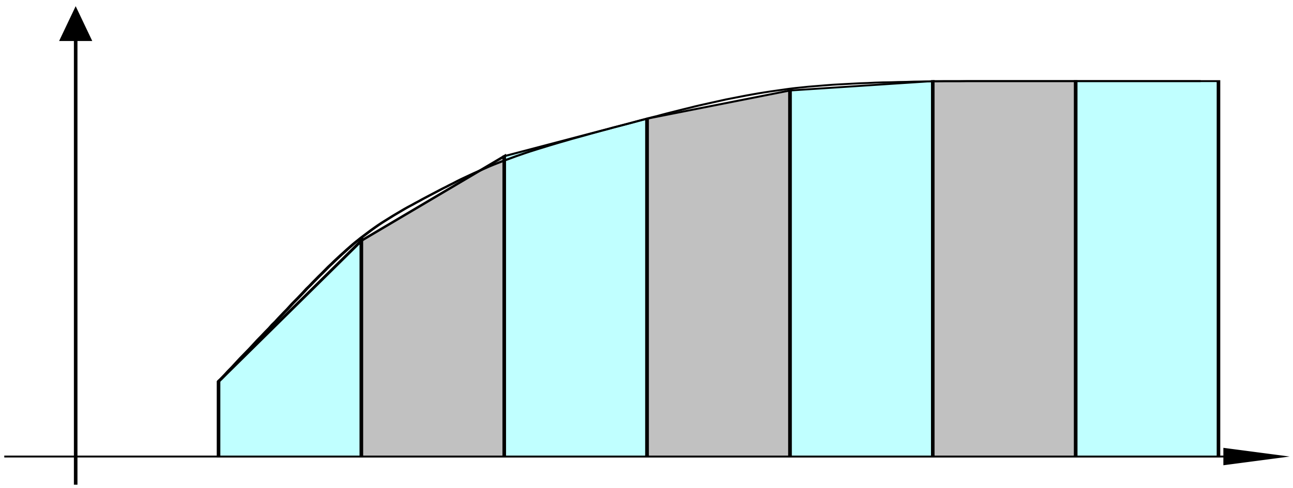
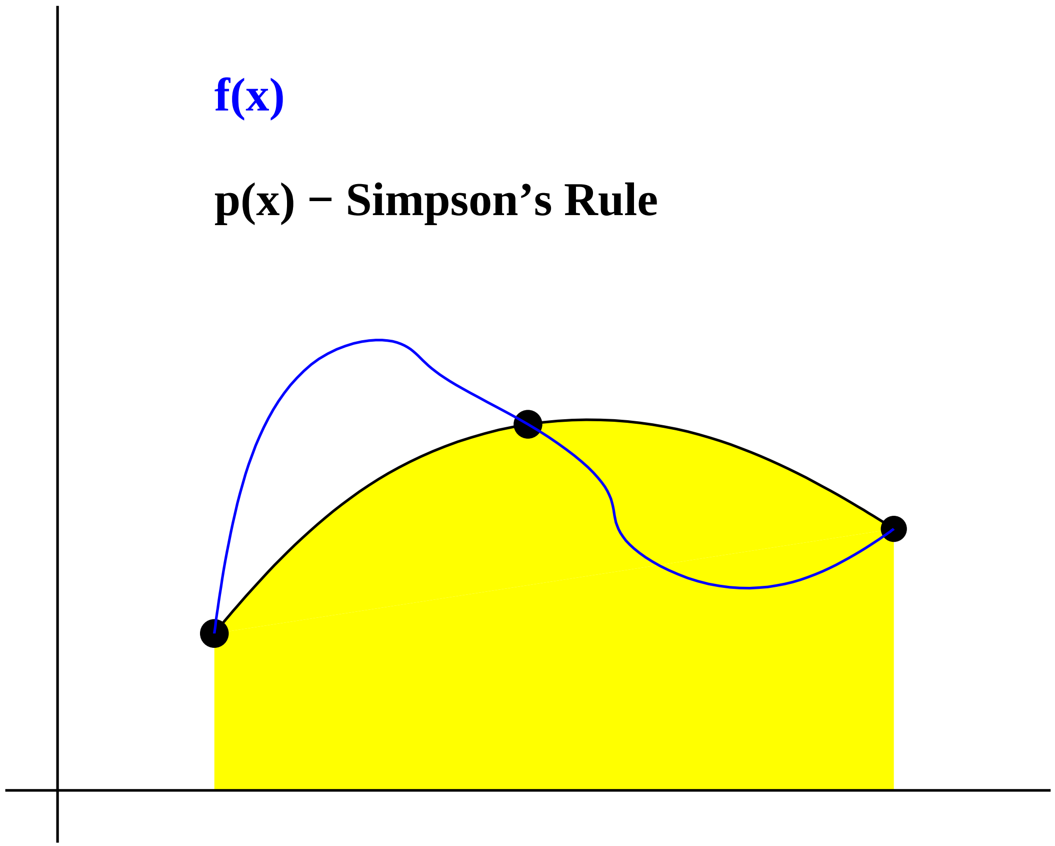
Last time: we worked hard (or did we just rely on Newton's hard work?) to find very accurate integration schemes; *i.e.* the *Newton-Cotes formulas.*

This time: Instead of working so hard — we use a low-order Newton-Cotes formula, but divide the integration interval into smaller sub-intervals.

> We use the sum of the integrations over the separate sub-intervals as the approximation to the whole integral.



$$\int_a^b f(x)\,dx = h\left[\frac{f(x_0) + 4f(x_1) + f(x_2)}{3}\right] - \frac{h^5}{90} f^{(4)}(\xi).$$



The exact solution:

$$\int_0^4 e^x dx = e^4 - e^0 = 53.59815$$

Simpson's Rule with $h = 2$

$$\int_0^4 e^x dx \approx \frac{2}{3}(e^0 + 4e^2 + e^4) = 56.76958.$$

The error is $\mathbf{-3.17143}$ (5.92%).

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Divide-and-Conquer: Simpson's Rule with $h = 1$

$$\int_0^2 e^x dx + \int_2^4 e^x dx \approx \frac{1}{3}(e^0 + 4e^1 + e^2) + \frac{1}{3}(e^2 + 4e^3 + e^4) = 53.86385$$

The error is $\mathbf{-0.26570}$. (0.50%) An improvement by a factor of 10!

The exact solution:

$$\int_0^4 e^x dx = e^4 - e^0 = 53.59815$$

Divide-and-Conquer: Simpson's Rule with $h = 1/2$

$$\int_0^1 + \int_1^2 + \int_2^3 + \int_3^4 e^x dx \approx \frac{1}{6}(e^0 + 4e^{1/2} + e^1) + \frac{1}{6}(e^1 + 4e^{3/2} + e^2)$$
$$+\frac{1}{6}(e^2 + 4e^{5/2} + e^3) + \frac{1}{6}(e^3 + 4e^{7/2} + e^4) = 53.61622$$

The error has been reduced to $\mathbf{-0.01807}$ (0.034%).

The exact solution:

$$\int_0^4 e^x dx = e^4 - e^0 = 53.59815$$

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h	abs-error	err/h	err/h^2	err/h^3	err/h^4
2	3.17143	1.585715	0.792857	0.396429	0.198214
1	0.26570	0.265700	0.265700	0.265700	0.265700
1/2	0.01807	0.036140	0.072280	0.144560	0.289120

Extending the table...

h	abs-error	err$/h$	err$/h^2$	err$/h^3$	**err$/h^4$**	err$/h^5$
2	3.171433	1.585716	0.792858	0.396429	**0.198215**	0.099107
1	0.265696	0.265696	0.265696	0.265696	**0.265696**	0.265696
1/2	0.018071	0.036142	0.072283	0.144566	**0.289132**	0.578264
1/4	0.001155	0.004618	0.018473	0.073892	**0.295566**	1.182266
1/8	0.000073	0.000580	0.004644	0.037152	**0.297215**	2.377716

Clearly, the **err$/h^4$** column seems to converge (to a non-zero constant) as $h \searrow 0$. The columns to the left seem to converge to zero, and the err$/h^5$ column seems to grow.

This is ***numerical evidence*** that the composite Simpson's rule has a convergence rate of $\mathcal{O}\left(h^4\right)$. *But, isn't Simpson's rule 5th order???*

For an even integer n: Subdivide the interval $[a, b]$ into n subintervals, and apply Simpson's rule on each consecutive pair of sub-intervals. With $h = (b - a)/n$ and $x_j = a + jh$, $j = 0, 1, \ldots, n$, we have

$$\int_a^b f(x)dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)dx$$

$$= \sum_{j=1}^{n/2} \left\{ \frac{h}{3} \Big[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j}) \Big] - \frac{h^5}{90} f^{(4)}(\xi_j) \right\},$$

for some $\xi_j \in [x_{2j-2}, x_{2j}]$, if $f \in C^4[a, b]$.

Since all the interior "even" x_{2j} points appear twice in the sum, we can simplify the expression a bit...

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) - f(x_n) + \sum_{j=1}^{n/2}\left[4f(x_{2j-1}) + 2f(x_{2j})\right]\right]$$
$$-\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j).$$

The error term is:

$$E(f) = -\frac{h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j)$$

If $f \in C^4[a,b]$, the **Extreme Value Theorem implies that** $f^{(4)}$ assumes its max and min in $[a,b]$. Now, since

$$\min_{x\in[a,b]} f^{(4)}(x) \le f^{(4)}(\xi_j) \le \max_{x\in[a,b]} f^{(4)}(x),$$

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$$\min_{x\in[a,b]} f^{(4)}(x) \leq \left[\frac{2}{n}\right] \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \max_{x\in[a,b]} f^{(4)}(x),$$

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$$\min_{x\in[a,b]} f^{(4)}(x) \leq \left[\frac{2}{n}\right] \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \leq \max_{x\in[a,b]} f^{(4)}(x),$$

By the **Intermediate Value Theorem** $\exists \mu \in (a,b)$ so that

$$f^{(4)}(\mu) = \frac{2}{n} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) \quad \Leftrightarrow \quad \frac{n}{2} f^{(4)}(\mu) = \sum_{j=1}^{n/2} f^{(4)}(\xi_j)$$

We can now rewrite the error term:

$$E(f) = -\frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j) = -\frac{h^5}{180} n f^{(4)}(\mu),$$

or, since $h = (b-a)/n \Leftrightarrow n = (b-a)/h$, we can write

$$E(f) = -\frac{(b-a)}{180} \mathbf{h^4} \mathbf{f^{(4)}}(\mu).$$

Hence ***Composite Simpson's Rule*** has **degree of accuracy 3** (since it is exact for polynomials up to order 3), and the error is proportional to $\mathbf{h^4}$ — ***Convergence Rate*** $\mathcal{O}\left(h^4\right)$.

Theorem: —

Let $f \in C^4[a,b]$, n be even, $h = (b-a)/n$, and $x_j = a + jh$, $j = 0, 1, \ldots, n$. There exists $\mu \in (a,b)$ for which the ***Composite Simpson's Rule*** for n subintervals can be written with its error term as

$$\boxed{\begin{aligned}\int_a^b f(x)dx &= \frac{h}{3}\left[f(a) - f(b) + \sum_{j=1}^{n/2}\left[4f(x_{2j-1}) + 2f(x_{2j})\right]\right] \\ &\quad - \frac{(b-a)}{180}h^4 f^{(4)}(\mu).\end{aligned}}$$

Note: $x_0 = a$, and $x_n = b$.

Given the end points a and b and an even positive integer n:

[1] $h = (b - a)/n$

[2] `ENDPTS = f(a)+f(b)`

`ODDPTS = 0`

`EVENPTS = 0`

[3] `FOR` $i = 1, \ldots, n-1$ — (interior points)

$x = a + i * h$

`if` i `is even: EVENPTS += f(x)`

`if` i `is odd:  ODDPTS  += f(x)`

`END`

[4] `INTAPPROX = h*(ENDPTS+2*EVENPTS+4*ODDPTS)/3`

Romberg Integration is the combination of the **Composite Trapezoidal Rule** (CTR)

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{j=1}^{n-1} f(x_j)\right] - \frac{(b-a)}{12}h^2 f''(\mu)$$

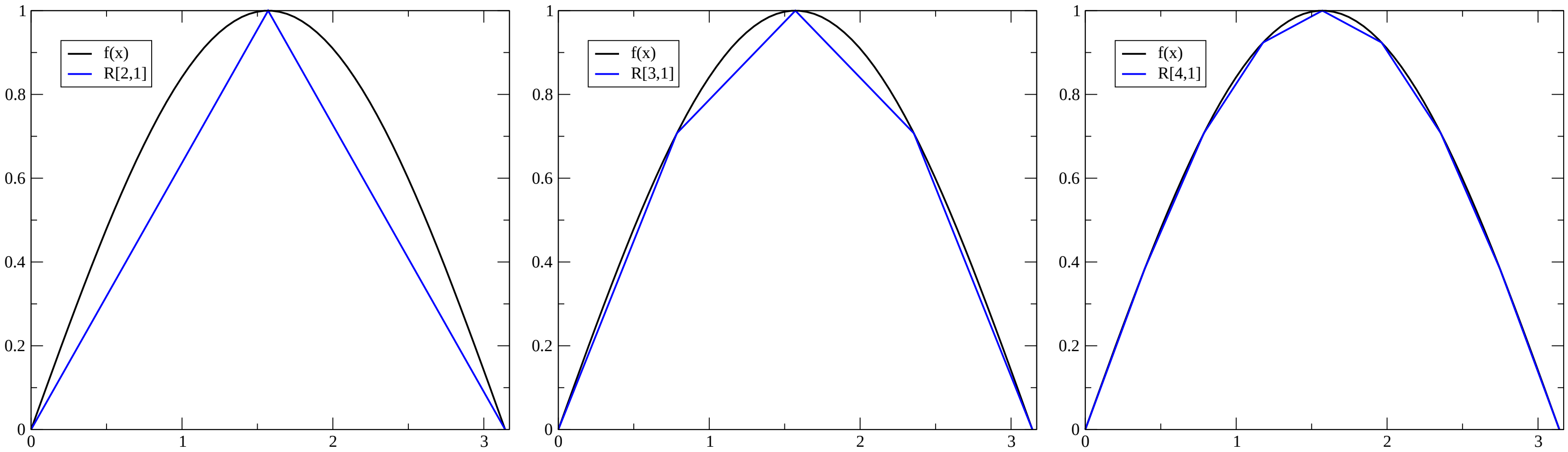
and **Richardson Extrapolation**.

Here, we know that the error term for regular Trapezoidal Rule is $\mathcal{O}(h^3)$. By the same argument as for Composite Simpson's Rule, this gets reduced to $\mathcal{O}(\mathbf{h^2})$ for the composite version.

Let $R_{k,1}$ denote the Composite Trapezoidal Rule with 2^{k-1} sub-intervals, and $h_k = (b-a)/2^{k-1}$. We get:

$$
\begin{aligned}
R_{1,1} &= \tfrac{h_1}{2}\left[f(a) + f(b)\right] \\
R_{2,1} &= \tfrac{h_2}{2}\left[f(a) + 2f(a+h_2) + f(b)\right] \\
&= \tfrac{(b-a)}{4}\left[f(a) + f(b) + 2f(a+h_2)\right] \\
&= \tfrac{1}{2}\left[R_{1,1} + h_1 f(a+h_2)\right] \\
&\vdots \\
R_{k,1} &= \underbrace{\frac{1}{2}\left[R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a+(2i-1)h_k)\right]}_{\text{Update formula, using previous value + new points}}
\end{aligned}
$$

Example: $R_{k,1}$ for $\int_0^\pi \sin(x)dx$



k	$R_{k,1}$
1	0
2	1.5707963267949
3	1.8961188979370
4	1.9742316019455
5	1.9935703437723
6	1.9983933609701
7	1.9995983886400

We know that the error term is $\mathcal{O}(h^2)$, so in order to eliminate this term we combine to consecutive entries $R_{k-1,1}$ and $R_{k,1}$ to form a higher order approximation $R_{k,2}$ of the integral.

$$\mathbf{R_{k,2} = R_{k,1} + \frac{R_{k,1} - R_{k-1,1}}{2^2 - 1}}$$

$R_{k,1} - \mathcal{O}\left(h^2\right)$	$R_{k,2}$
0	0
1.5707963267949	2.09439510239
1.8961188979370	2.00455975498
1.9742316019455	2.00026916994
1.9935703437723	2.00001659104
1.9983933609701	2.00000103336
1.9995983886400	2.00000006453

It turns out (Taylor expand to check) that the complete error term for the Trapezoidal rule only has even powers of h:

$$\int_a^b f(x) = R_{k,1} - \sum_{i=1}^{\infty} E_{2i} h_k^{2i}.$$

Hence the $R_{k,2}$ approximations have error terms that are of size $\mathcal{O}(\mathbf{h^4})$.

To get $\mathcal{O}(h^6)$ approximations, we compute

$$\mathbf{R_{k,3} = R_{k,2} + \frac{R_{k,2} - R_{k-1,2}}{4^2 - 1}}$$

In general, since we only have even powers of h in the error expansion:

$$\mathbf{R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}}$$

Revisiting $\int_0^\pi \sin(x)dx$:

$R_{k,1}$ — $\mathcal{O}\left(h^2\right)$	$R_{k,2}$ — $\mathcal{O}\left(h^4\right)$	$R_{k,3}$ — $\mathcal{O}\left(h^6\right)$	$R_{k,4}$ — $\mathcal{O}\left(h^8\right)$
0			
1.570796326794897	2.094395102393195		
1.896118897937040	2.004559754984421	1.998570731823836	
1.974231601945551	2.000269169948388	1.999983130945986	2.000005549979671
1.993570343772340	2.000016591047935	1.999999752454572	2.000000016288042
1.998393360970145	2.000001033369413	1.999999996190845	2.000000000059674
1.999598388640037	2.000000064530001	1.999999999940707	2.000000000000229

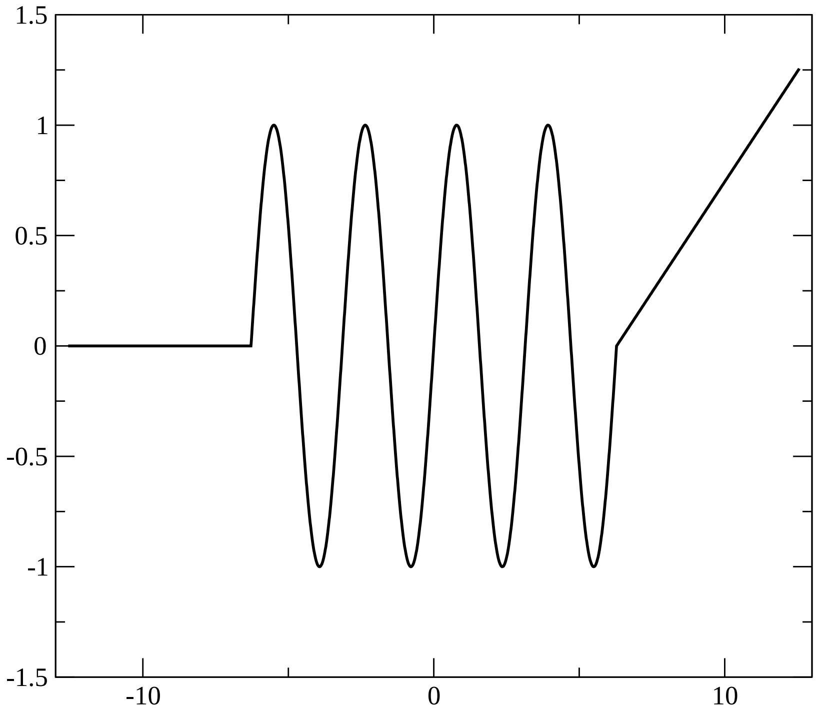
Homework? No, enough already — Here's the code outline!

```
% Romberg Integration for sin(x) over [0,pi]
a = 0; b = pi; % The Endpoints
R = zeros(7,7);
R(1,1) = (b-a)/2*(sin(a)+sin(b));
for k = 2:7
  h = (b-a)/2^(k-1);
  R(k,1)=1/2*(R(k-1,1)+2*h*sum(sin(a+(2*(1:(2^(k-2)))-1)*h)));
end
for j = 2:7
  for k = j:7
   R(k,j) = R(k,j-1)+(R(k,j-1)-R(k-1,j-1))/(4^(j-1)-1);
  end
end
disp(R)
```

Adaptive and Gaussian Quadrature

The ***composite formulas*** require ***equally spaced nodes***.

This is not good if the function we are trying to integrate has both regions with large fluctuations, and regions with small variations.



We need many points where the function fluctuates, but few points where it is close to constant or linear.

Idea: Cleverly predict (or measure) the amount of variation and automatically add more points where needed.

We are going to discuss this in the context of Composite Simpson's rule, but the approach can be adopted for other integration schemes.

First we are going to develop a way to ***measure the error*** — a numerical estimate of the actual error in the numerical integration. Note that here just knowing the structure of the error term is not enough! (We will however use the structure of the error term in our derivation of the numerical error estimate.)

Then we will use the error estimate to decide whether to accept the value from CSR, or if we need to refine further (recompute with smaller h).

Notation — "One-step" Simpson's Rule:

$$\int_a^b f(x)\,dx = S(f;\, a, b) - \underbrace{\frac{h_1^5}{90} f^{(4)}(\mu_1)}_{\mathbf{E}(\mathbf{f};\,\mathbf{h_1},\mu_\mathbf{1})}, \quad \mu_1 \in (a, b),$$

where

$$S(f;\, a, b) = \frac{(b-a)}{6}\left[f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)\right], \quad h_1 = \frac{(b-a)}{2}.$$

With this notation, we can write CSR with $n = 4$, and $h_2 = (b - a)/4 = h_1/2$:

$$\int_a^b f(x)\,dx = S(f;\,a,\,\tfrac{a+b}{2}) + S(f\,;\tfrac{a+b}{2},b) - E(f;h_2,\mu_2).$$

We can squeeze out an estimate for the error by noticing that

$$E(f;h_2,\mu_2) = \frac{1}{16}\left(\frac{h_1^5}{90} f^{(4)}(\mu_2)\right) = \frac{1}{16} E(f;h_1,\boldsymbol{\mu_2}).$$

Now, **assuming** $f^{(4)}(\mu_1) \approx f^{(4)}(\mu_2)$, we do a little bit of algebra magic with our two approximations to the integral...

$$E(f;h_2,\mu_2) = \frac{1}{32}\left(\frac{h_1^5}{90}f^{(4)}(\mu_2^1)\right) + \frac{1}{32}\left(\frac{h_1^5}{90}f^{(4)}(\mu_2^2)\right)$$

where $\mu_2^1 \in [a, \frac{a+b}{2}]$, $\mu_2^2 \in [\frac{a+b}{2}, b]$.

If $f \in C^4[a,b]$, then we can use our old friend, the **intermediate value theorem**:

$$\exists \mu_2 \in [\mu_2^1, \mu_2^2] \subset [a,b] : \; f^{(4)}(\mu_2) = \frac{f^{(4)}(\mu_2^1) + f^{(4)}(\mu_2^2)}{2}.$$

So it follows that

$$E(f;h_2,\mu_2) = \frac{1}{16}\left(\frac{h_1^5}{90}f^{(4)}(\mu_2)\right)$$

Now we have

$$S(f;\,a,\tfrac{a+b}{2}) + S(f;\,\tfrac{a+b}{2},b) - \frac{1}{16}\left(\frac{h_1^5}{90}f^{(4)}(\mu_2)\right)$$
$$= S(f;\,a,b) - \frac{h_1^5}{90}f^{(4)}(\mu_1).$$

Now use the assumption $f^{(4)}(\mu_1) \approx f^{(4)}(\mu_2)$ (and replace μ_1 and μ_2 by μ):

$$\mathbf{\frac{h_1^5}{90}f^{(4)}}(\mu) \approx \frac{16}{15}\Big[S(f;\,a,b) - S(f;\,a,(a+b)/2) - S(f;\,(a+b)/2,b)\Big],$$

notice that $\frac{h_1^5}{90}f^{(4)}(\mu) = E(f;\,h_1,\mu) = 16E(f;\,h_2,\mu)$. Hence

$$E(f;\,h_2,\mu) \approx \frac{1}{15}\Big[S(f;\,a,b) - S(f;\,a,(a+b)/2) - S(f;\,(a+b)/2,b)\Big],$$

Using the estimate of $\frac{h_1^5}{90} f^{(4)}(\mu)$, we have

$$\boxed{\begin{aligned}&\left|\int_a^b f(x)dx - S(f;\, a, (a+b)/2) - S(f;\, (a+b)/2, b)\right| \\ &\quad \approx \frac{1}{15}\left|S(f;\, a, b) - S(f;\, a, (a+b)/2) - S(f;\, (a+b)/2, b)\right|\end{aligned}}$$

Notice!!! $S(f;\, a, (a+b)/2) + S(f;\, (a+b)/2, b)$ approximates $\int_a^b f(x)dx$ **15 times better** than it agrees with the known quantity $S(f;\, a, b)$!!!

We will apply Simpson's rule to

$$\int_0^{\pi/2} \sin(x)\, dx = 1.$$

Here,

$$\begin{aligned}
\mathbb{S}_1(\sin(x);\, 0, \pi/2) &= S(\sin(x);\, 0, \pi/2) \\
&= \frac{\pi}{12}\left[\sin(0) + 4\sin(\pi/4) + \sin(\pi/2)\right] = \frac{\pi}{12}\left[2\sqrt{2} + 1\right] \\
&= 1.00227987749221.
\end{aligned}$$

$$\begin{aligned}
\mathbb{S}_2(\sin(x);\, 0, \pi/2) &= S(\sin(x);\, 0, \pi/4) + S(\sin(x);\, \pi/4, \pi/2) \\
&= \frac{\pi}{24}\left[\sin(0) + 4\sin(\pi/8) + 2\sin(\pi/4) + 4\sin(3\pi/8) + \sin(\pi/2)\right] \\
&= 1.00013458497419.
\end{aligned}$$

The error estimate is given by

$$\frac{1}{15}\left[\mathbb{S}_1(\sin(x);\,0,\pi/2) - \mathbb{S}_2(\sin(x);\,0,\pi/2)\right]$$
$$= \frac{1}{15}\left[1.00227987749221 - 1.00013458497419\right] = 0.00014301950120.$$

This is a very good approximation of the actual error, which is 0.00013458497419.

OK, we know how to get an error estimate. How do we use this to create an adaptive integration scheme???

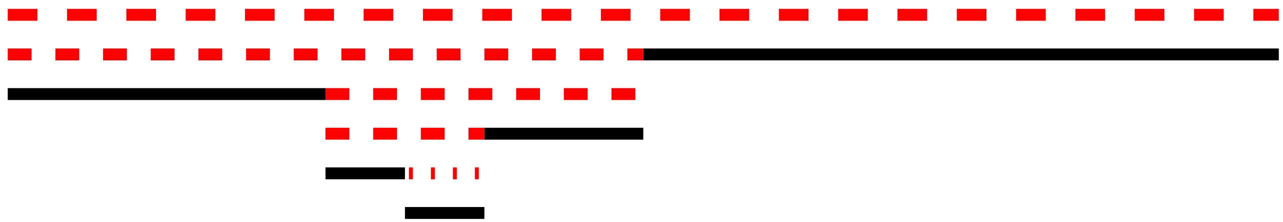
We want to approximate $\mathcal{I} = \int_a^b f(x)\,dx$ with an error less than ϵ (a specified tolerance).

[1] Compute the two approximations
$\mathbb{S}_1(f(x);\, a, b) = S(f(x);\, a, b)$, and
$\mathbb{S}_2(f(x);\, a, b) = S(f(x);\, a, \frac{a+b}{2}) + S(f(x);\, \frac{a+b}{2}, b)$.

[2] Estimate the error, if the estimate is less than ϵ, we are done. Otherwise...

[3] Apply steps **[1]** and **[2]** recursively to the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ with tolerance $\epsilon/2$.

Adaptive Quadrature, Interval Refinement Example #1



The funny figure above is supposed to illustrate a possible sub-interval refinement hierarchy. Red dashed lines illustrate failure to satisfy the tolerance, and **black** lines illustrate satisfied tolerance.

level	tol	interval
1	ϵ	$[\mathbf{a}, \mathbf{b}]$
2	$\epsilon/2$	$[\mathbf{a}, \mathbf{a} + \frac{\mathbf{b}-\mathbf{a}}{\mathbf{2}}]$ $[\mathbf{a} + (\mathbf{b} - \mathbf{a})/\mathbf{2}, \mathbf{b}]$
3	$\epsilon/4$	$[\mathbf{a}, \mathbf{a} + \frac{\mathbf{b}-\mathbf{a}}{\mathbf{4}}]$ $[\mathbf{a} + \frac{\mathbf{b}-\mathbf{a}}{\mathbf{4}}, \mathbf{a} + \frac{\mathbf{b}-\mathbf{a}}{\mathbf{2}}]$
$\vdots$		

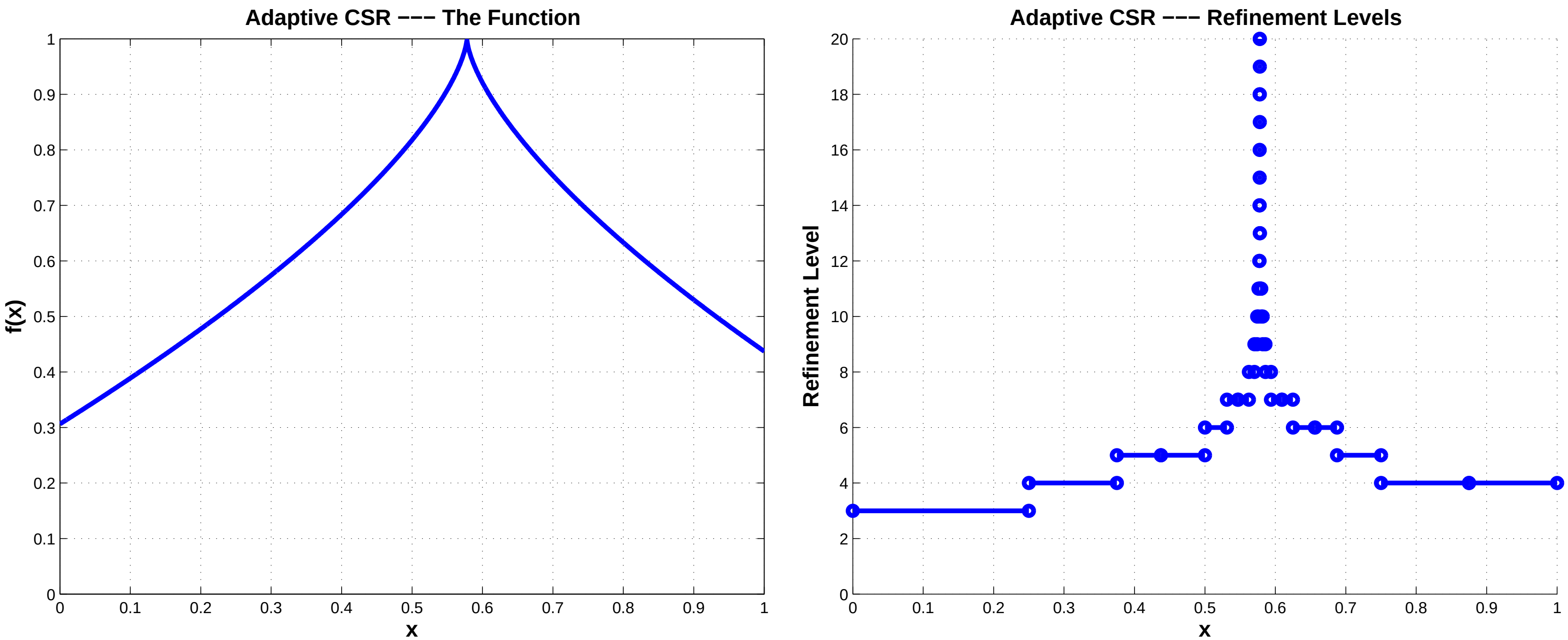


Figure: Application of adaptive CSR to the function $f(x) = 1 - \sqrt[3]{(x - \frac{\pi}{2e})^2}$. Here, we have required that the estimated error be less than 10^{-6}. The left panel shows the function, and the right panel shows the number of refinement levels needed to reach the desired accuracy. At completion we have the value of the integral being 0.61692712, with an estimated error of $3.93 \cdot 10^{-7}$.

Idea: Evaluate the function at a set of **optimally chosen** points in the interval.

We will choose $\{x_0, x_1, \ldots, x_n\} \in [a, b]$ and coefficients c_i, so that the approximation

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i f(x_i)$$

is exact for the largest class of polynomials possible.

We have already seen that the open Newton-Cotes formulas sometime give us better "bang-for-buck" than the closed formulas (*e.g.* the mid-point formula uses only 1 point and is as accurate as the two-point trapezoidal rule). — Gaussian quadrature takes this one step further.

	Newton-Cotes		Gaussian
	Open	Closed	
Quadrature Points	Degree of Accuracy	Degree of Accuracy	Degree of Accuracy
1	**1***	—	1
2	1	1	3
3	3	**3#**	5
4	3	3	7
5	5	5	9

* — The mid-point rule.

— Simpson's rule.

The mid-point rule is the only optimal scheme we have see so far.

Suppose we want to find an optimal two-point formula:

$$\int_{-1}^{1} f(x)\,dx = c_1 f(x_1) + c_2 f(x_2).$$

Since we have 4 parameters to play with, we can generate a formula that is ***exact up to polynomials of degree 3***. We get the following 4 equations:

$$\begin{array}{rclcl|rcl}
\int_{-1}^{1} 1\,dx & = & 2 & = & c_1 + c_2 & c_1 & = & 1 \\
\int_{-1}^{1} x\,dx & = & 0 & = & c_1 x_1 + c_2 x_2 & c_2 & = & 1 \\
\int_{-1}^{1} x^2\,dx & = & \frac{2}{3} & = & c_1 x_1^2 + c_2 x_2^2 & x_1 & = & -\frac{\sqrt{3}}{3} \\
\int_{-1}^{1} x^3\,dx & = & 0 & = & c_1 x_1^3 + c_2 x_2^3 & x_2 & = & \frac{\sqrt{3}}{3}
\end{array}$$

We could obtain higher order formulas by adding more points, computing the integrals, and solving the resulting non-linear system of equations... but it gets very painful, very fast.

The **Legendre Polynomials** come to our rescue!

The Legendre polynomials $P_n(x)$ are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1$, *i.e.*

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \alpha_n \delta_{n,m} = \begin{cases} 0 & m \neq n \\ \alpha_n & m = n \end{cases}.$$

If $P(x)$ is a polynomial of degree less than n, then

$$\int_{-1}^{1} P_n(x) P(x)\, dx = 0.$$

We will see Legendre polynomials in more detail later. For now, all we need to know is that they satisfy the property

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \alpha_n \delta_{n,m}.$$

and the first few Legendre polynomials are

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= x^2 - 1/3 \\
P_3(x) &= x^3 - 3x/5 \\
P_4(x) &= x^4 - 6x^2/7 + 3/35 \\
P_5(x) &= x^5 - 10x^3/9 + 5x/21.
\end{aligned}$$

It turns out that the roots of the Legendre polynomials are the nodes in Gaussian quadrature.

Theorem: — Suppose that $\{x_1, x_2, \ldots, x_n\}$ are the roots of the n^{th} Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$, the coefficients c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\, dx.$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)\, dx = \sum_{i=1}^{n} c_i P(x_i).$$

Let us first consider a polynomial, $P(x)$ with degree less than n. $P(x)$ can be rewritten as an $(n-1)$-st Lagrange polynomial with nodes at the roots of the n^{th} Legendre polynomial $P_n(x)$. This representation is exact since the error term involves the n^{th} derivative of $P(x)$, which is zero. Hence,

$$\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} \left[\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} P(x_j) \right] dx$$

$$= \sum_{i=1}^{n} \left[\int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\mathbf{x} - \mathbf{x_j}}{\mathbf{x_i} - \mathbf{x_j}} \,\mathbf{dx} \right] P(x_j) = \sum_{i=1}^{n} \mathbf{c_i} P(x_i),$$

which verifies the result for polynomials of degree less than n.

If the polynomial $P(x)$ of degree $[n, 2n)$ is divided by the n^{th} Legendre polynomial $P_n(x)$, we get:

$$P(x) = Q(x)P_n(x) + R(x)$$

where both $Q(x)$ and $R(x)$ are of degree less than n.

[1] Since $\deg(Q(x)) < n$

$$\int_{-1}^{1} Q(x)P_n(x)\,dx = 0.$$

[2] Further, since x_i is a root of $P_n(x)$:

$$P(x_i) = Q(x_i)P_n(x_i) + R(x_i) = R(x_i).$$

[3] Now, since $\deg(R(x)) < n$, the first part of the proof implies

$$\int_{-1}^{1} R(x)\,dx = \sum_{i=1}^{n} c_i R(x_i).$$

Putting **[1]**, **[2]** and **[3]** together we arrive at

$$\begin{aligned}\int_{-1}^{1} P(x)\,dx &= \int_{-1}^{1} [Q(x)P_n(x) + R(x)]\,dx \\ &= \int_{-1}^{1} R(x)\,dx = \sum_{i=1}^{n} c_i R(x_i) \\ &= \sum_{i=1}^{n} c_i P(x_i),\end{aligned}$$

which shows that the formula is exact for all polynomials $P(x)$ of degree less than $2n$. $\square$

By a simple linear transformation,

$$t = \frac{2x - a - b}{b - a} \quad \Leftrightarrow \quad x = \frac{(b-a)t + (b+a)}{2},$$

we can apply the Gaussian Quadrature formulas to any interval

$$\int_a^b f(x)\, dx = \int_{-1}^{1} f\left(\frac{(b-a)t + (b+a)}{2}\right) \frac{(b-a)}{2} dt.$$

Degree	$P_n(x)$	Roots / Quadrature points
2	$x^2 - 1/3$	$\{-1/\sqrt{3},\ 1/\sqrt{3}\}$
3	$x^3 - 3x/5$	$\{-\sqrt{3/5},\ 0,\ \sqrt{3/5}\}$
4	$x^4 - 6x^2/7 + 3/35$	$\{-0.86114,\ -0.33998,\ 0.33998,\ 0.86114\}$

$$\int_0^{\pi/4} (\cos(x))^2\, dx = \frac{1}{4} + \frac{\pi}{8} = 0.642699081698724$$

Degree	Quadrature points	Coefficients
2	0.16597, 0.61942	1, 1
3	0.08851, 0.39270, 0.69688	0.55556, 0.88889, 0.55556
4	0.05453, 0.25919, 0.52621, 0.73087	0.34785, 0.65215, 0.65215, 0.34785

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Degree	Integral approximation	Error
2	0.642317235049753	0.0003818466489...
3	0.642701112090729	0.0000020303920...
4	0.642699075999924	0.0000000056988...