1. a. Consider an animal that lives four years and reproduces annually. Animals that are 0-1 years old don't reproduce and only 40% ($s_1 = 0.4$) of them survive to the next year. Animals that are 1-2 years old produce on average $b_2 = 1.5$ offspring and 70% ($s_2 = 0.7$) of them survive to the next year. Animals 2-3 years old produce on average $b_3 = 2.2$ offspring and 75% ($s_3 = 0.75$) of them survive to the next year. Finally, animals 3-4 years old produce $b_4 = 3.4$ offspring. Create a model using a Leslie matrix, L, of the form:

$$P_{n+1} = LP_n.$$

Find the steady-state percentage of each age group. Determine how long it takes for this population to double after it has reached its steady-state distribution.

b. Assume that a fraction of 2-4 year olds are harvested. That is, the survival rates s_2 and s_3 are reduced. If the survival rates are reduced by a fraction α, so that the survival rate of 1-2 year olds is $0.7\,\alpha$ and the survival rate of 2-3 year olds is $0.75\,\alpha$. Determine the value of α that leaves the population at a constant value. For this value of α (to at least 3 significant figures), if there are 550 mature (3-4 year olds), then determine the total population and number in each population age group. How many animals are harvested annually under these conditions?

2. One problem with the Leslie matrix models is that they are linear and so result in either exponentially growing or exponentially declining populations. One modification adds a logistic growth factor, creating a nonlinear model, known as the Leslie matrix model with density-dependent recruitment (LMMDDR). The model is given by

$$X(n+1) = q(x(n))LX(n),$$

where L (an $m \times m$ matrix) has a dominant eigenvalue $\lambda_1 > 1$ and q satisfies

$$q(x(n)) = \frac{K}{K + (\lambda_1 - 1)x(n)},$$

where K is the carrying capacity and $x(n) = \sum_{i=1}^{m} x_i(n)$, with $X(n) = (x_1(n), x_2(n), \ldots x_m(n))^T$.

a. Suppose that

$$L = \begin{pmatrix} 0 & 2.95 & 4.1 \\ 0.5 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix}$$

and $K = 100$. Find the dominant eigenvalue and associated eigenvector for L. Write the LMMDDR model.

b. Begin with a population $X(0) = (5, 5, 5)^T$, then simulate the model for 20 generations. Graph the populations of each age group and the total population. Determine the limiting populations in each of the age classes for this population model. Give a brief biological description of this model.

3. Since 1973, the British Forestry Commission has surveyed for the presence of the American gray squirrel (*Sciurus carolinensis Gmelin*) and the native red squirrel (*Sciurus vulgaris L.*).

From two consecutive years of data for 10 km square regions across Great Britain, data were collected on movement of the two types of squirrels. The transition matrix for red squirrels, gray squirrels, both, or neither in that order was given by

$$T = \begin{pmatrix} 0.8797 & 0.0382 & 0.0527 & 0.0008 \\ 0.0212 & 0.8002 & 0.0041 & 0.0143 \\ 0.0981 & 0.0273 & 0.8802 & 0.0527 \\ 0.0010 & 0.1343 & 0.0630 & 0.9322 \end{pmatrix}.$$

Find the equilibrium distribution of squirrels based on this transition matrix. Does this model suggest that the invasive gray species will significantly displace the native red squirrel over long periods of time?

4. In lecture, we saw that a stochastic birth only process could be given by the differential equation

$$\frac{dP_n}{dt} + \lambda N P_n = \lambda(N-1)P_{N-1},$$

with

$$P_N(0) = \begin{cases} 0 & N \neq N_0 \\ 1 & N = N_0 \end{cases},$$

where $P_N(t)$ is the probability that there are exactly N individuals in a population. The solution was given by

$$\begin{aligned} P_{N_0}(t) &= e^{-\lambda N_0 t} \\ P_{N_0+1}(t) &= N_0 e^{-\lambda N_0 t}(1-e^{-\lambda t}) \\ &\dots \\ P_{N_0+j}(t) &= \frac{N_0(N_0+1)\cdots(N_0+j-1)}{j!} e^{-\lambda N_0 t}(1-e^{-\lambda t})^j. \end{aligned}$$

Use mathematical induction to prove that this last formula holds for all j.

5. The expected population at time t, $E(t)$, is obtained from the formula

$$E(t) = \sum_{j=0}^{\infty}(N_0+j)P_{N_0+j}(t).$$

a. Can you explain why this is valid using probabilistic ideas?

b. Show that $E(t) = N_0 e^{\lambda t}$. [Hint: Explicitly calculate the first few terms of the summation; compare this to the derivative of the binomial expansion.]

c. Explain the significance of Part b.

6. Consider the equation above for $P_{N_0+j}(t)$. Assume that $N_0 = 10,000$. Estimate λ, if it is observed that a total of 4500 births occur in 20 days. [Hint: See the previous exercise.]

7. Consider the following definite integral:

$$\int_{-2}^{5} 4e^{-0.2x^2}\,dx.$$

Find the exact value of this integral. Use the Monte Carlo method discussed in class to simulate this integral with 10,000 random numbers. Compute the average and standard deviation for 100 Monte Carlo simulations of this integral.