1. In 1961, a simplified model for the complete Hodgkin and Huxley model for the giant squid axon was introduced by FitzHugh and Nagumo. This model assumed the sodium current was fast, so set it in quasi-steady state and assumed the leakage current was effectively constant. They also simplified the current functions, so that the only nonlinear function is a cubic. This reduced the Hodgkin-Huxley four dimensional model to the simpler two dimensional model, which after a reparameterization becomes:

$$\begin{aligned} \frac{dv}{dt} &= v(a-v)(v-1) - w + I_a, \\ \frac{dw}{dt} &= bv - \gamma w, \end{aligned}$$

where a, I_a, b, and γ are parameters for the FitzHugh-Nagumo model (FHN). The variable v represents the membrane potential, and w represents the current of potassium ions. The parameter I_a is the externally applied clamp voltage.

a. Draw the nullclines and show both the phase portrait (v and w) and time series ($v(t)$ for $t \in [0, 200]$ with $w(0) = 0$ and $v(0) = 0.1$) for the FHN model with $a = 0.2$, $I_a = 0.05$, and $b = \gamma = 0.002$. List all equilibria. Find the eigenvalues for the equilibria, and classify each equilibrium (for example, saddle node). Repeat this process for an applied external voltage of $I_a = 0.15$. Discuss briefly what each of these simulations is showing in terms of the membrane potential and the potassium currents.

b. Find the smallest applied external voltage $I_a > 0$ that produces a Hopf bifurcation. Show a time series simulation ($v(t)$ for $t \in [0, 200]$ with $w(0) = 0$ and $v(0) = 0.1$) for this value of I_a and give the value of the equilibrium and its eigenvalues.

c. As voltage is continually applied to an experimental sample, the membrane channels degrade. Since γ represents the loss of potassium ions, then assume that as more experiments are performed, the channels degrade and γ increases. Assume that $a = 0.2$, $I_a = 0.05$, and $b = 0.002$. Vary γ from 0.002 to 0.01 and describe all bifurcations that occur (Hopf bifurcations, reverse Hopf bifurcations, saddle node bifurcations with new equilibria, etc.). Give the values where these bifurcations occur and describe what happens qualitatively to the solution of the FHN model at each bifurcation.