1. a. This is a separable differential equation, so
\[
\frac{dy}{dt} = 3t^2 y
\]
\[
\int \frac{dy}{y} = \int 3t^2 dt
\]
\[
\ln |y| = t^3 + C
\]
\[
y(t) = e^{t^3+C} = Ae^{t^3}
\]
c. This is a linear equation like Newton’s Law of Cooling. We write \( \frac{dy}{dt} = 3 - 2y = -2 \left( y - \frac{3}{2} \right) \), so take \( z(t) = y(t) - \frac{3}{2} \). Thus, \( z' = -2z \), which has the solution
\[
z(t) = Ce^{-2t} = y(t) - \frac{3}{2}
\]
\[
y(t) = Ce^{-2t} + \frac{3}{2}
\]
e. This is a separable differential equation, so
\[
\frac{dy}{dt} = 2ty^2
\]
\[
\int y^{-2} dy = \int 3t^2 dt
\]
\[
-y^{-1} = t^3 + C
\]
\[
y(t) = -\frac{1}{t^3 + C}
\]
g. This is a separable differential equation, so
\[
2ty \frac{dy}{dt} = t^2 + 4
\]
\[
\int 2y dy = \int \left( t + \frac{4}{t} \right) dt
\]
\[
y^2 = \frac{t^2}{2} + 4 \ln |t| + C
\]
\[
y(t) = \sqrt{\frac{t^2}{2} + 4 \ln |t| + C}
\]
2. a. This is a separable differential equation, so
\[
\frac{dy}{dt} = 2ty
\]
\[
\int \frac{dy}{y} = \int 2tdt
\]
\[
\ln |y| = t^2 + C
\]
\[
y(t) = e^{t^2+C} = Ae^{t^2}
\]
From the initial condition, \( y(0) = 5 = A \), it follows that the solution is given by

\[ y(t) = 5e^{t^2}. \]

c. This is a separable differential equation, so

\[
(1 + 2y) \frac{dy}{dt} = 2t
\]

\[
\int (1 + 2y)dy = \int 2tdt
\]

\[ y + y^2 = t^2 + C \]

This gives an implicit answer for the solution into which we substitute the initial condition, \( y(2) = 0 \). It follows that

\[ 0 + 0^2 = 2^2 + C \quad \text{or} \quad C = -4. \]

The implicit answer is

\[ y^2(t) + y(t) = t^2 - 4, \]

which can be solved explicitly to give

\[ y(t) = \frac{-1 + \sqrt{4t^2 - 15}}{2}. \]

f. This is a separable differential equation, so

\[
t \frac{dy}{dt} = 2y
\]

\[
\int \frac{dy}{y} = \int \frac{2dt}{t}
\]

\[ \ln |y| = 2 \ln |t| + C \]

\[ y(t) = e^{2\ln|t|+C} = e^{\ln(t^2)}e^C = At^2 \]

The initial condition, \( y(1) = 4 \), is substituted into this equation giving \( y(1) = 4 = A(1)^2 \) or \( A = 4 \). It follows that

\[ y(t) = 4t^2. \]

g. This is a separable differential equation, so

\[
\frac{dy}{dt} = 2 \cos(2t)y^2
\]

\[ \int \frac{dy}{y^2} = \int 2 \cos(2t)dt \]

\[ -\frac{1}{y(t)} = \sin(2t) + C \]

\[ y(t) = -\frac{1}{\sin(2t) + C} \]

The initial condition, \( y(0) = 1 \), is substituted into this equation giving \( y(0) = 1 = -\frac{1}{C} \) or \( C = -1 \). It follows that

\[ y(t) = \frac{1}{1 - \sin(2t)}. \]
4. a. Begin by separating the variables in this differential equation. The result is as follows:

\[
\frac{dV}{dt} = 0.04V^{3/4}
\]

\[
\int V^{-3/4}dV = \int 0.04dt
\]

\[
4V^{1/4} = 0.04t + C
\]

\[
V(t) = \left(\frac{0.04t + C}{4}\right)^4
\]

The initial condition, \(V(0) = 1\), is substituted into this equation giving \(V(0) = 1 = \left(\frac{0}{4}\right)^4\) or \(C = 4\). It follows that

\[
V(t) = (0.04t + 1)^4
\]

b. For the cell to double its volume, we solve

\[
V(t) = (0.04t + 1)^4 = 2
\]

\[
(0.04t + 1) = 2^{1/4}
\]

\[
t = 100\left(2^{1/4} - 1\right) \approx 18.92.
\]

Thus, it takes about 18.92 time units for the cell to double its volume.

5. a. The differential equation described in this problem is given by

\[
\frac{dV}{dt} = kV^{2/3}
\]

The general solution is found by separating variables to give

\[
\int V^{-2/3}dV = \int kdt
\]

\[
3V^{1/3} = kt + C
\]

\[
V(t) = \left(\frac{kt + C}{3}\right)^3
\]

b. If \(V(0) = 1\), then \(V(0) = \left(\frac{C}{3}\right)^3 = 1\) implies that \(C = 3\). Thus, the solution for the growth of the raindrop is

\[
V(t) = \left(\frac{0.1t + 3}{3}\right)^3
\]

For this solution to grow to 8 units,

\[
V(t) = \left(\frac{0.1t + 3}{3}\right)^3 = 8 \quad \text{or} \quad \frac{0.1t + 3}{3} = 2.
\]

It follows that \(t = 30\) time units.
7. a. Since \(N(t) = k\pi x^2(t), \ x(t) = \sqrt{N(t)/k\pi}\) It follows that \(C(t)\) can be written

\[
C(t) = 2\sqrt{\frac{\pi N(t)}{k}}.
\]

b. The assumption is that the rate of spread of the disease (which is the change in number of diseased trees) is proportional to the circumference of the infected region. Since \(C(t)\) is the circumference (and we chose \(q\) to be the proportionality factor), the differential equation is given by

\[
\frac{dN}{dt} = qC, \quad N(0) = 1,
\]
assuming we start with one infected tree in the middle of the grove.

c. With the model for the spread of a disease in an orchard, we apply the separation of variables technique to the initial value problem

\[
\frac{dN}{dt} = 2q\sqrt{\frac{\pi}{k}} N^{1/2}, \quad N(0) = 1.
\]

It follows that

\[
\int N^{-1/2}dV = 2q\sqrt{\frac{\pi}{k}} dt
\]

\[
2N^{1/2} = 2q\sqrt{\frac{\pi}{k}} t + C
\]

\[
N(t) = \left(q\sqrt{\frac{\pi}{k}} t + \frac{C}{2}\right)^2.
\]

The initial condition, \(N(0) = 1\), is substituted into this equation giving \(N(0) = 1 = (\frac{C}{2})^2\) or \(C = 2\). It follows that

\[
N(t) = \left(q\sqrt{\frac{\pi}{k}} t + 1\right)^2.
\]

8. a. The solution to the Malthusian growth equation is \(Y(t) = 2000e^{0.08t}\). It doubles when \(Y(t) = 4000\), so \(4000 = 2000e^{0.08t}\) or \(e^{0.08t} = 2\). Thus, \(0.08t = \ln(2)\) or \(t = 12.5\ln(2) = 8.66\text{ hr.}\)

b. This is a separable equation, so

\[
\frac{dY}{dt} = (0.08 - 0.002 t)Y
\]

\[
\int \frac{dY}{Y} = \int (0.08 - 0.002 t)dt
\]

\[
\ln |Y(t)| = 0.08 t - 0.001 t^2 + C
\]

\[
Y(t) = \exp 0.08 t - 0.001 t^2 + C = A e^{0.08 t - 0.001 t^2}
\]

The initial condition gives \(Y(0) = 2000 = A\). It follows that

\[
Y(t) = 2000 e^{0.08 t - 0.001 t^2}.
\]
c. The maximum occurs when the \( \frac{dY}{dt} = 0 \), which is true when \( 0.08 - 0.002t = 0 \) or \( t = 40 \). From the equation above, \( Y(40) = 2000e^{0.08(40) - 0.001(40)^2} = 2000e^{1.6} \approx 9906 \). The population is 2000 when the exponent of the solution is zero, so \( 0.08 - 0.001t^2 = 0 \) when \( t = 0 \) or 80 hrs. Thus, the population returns to 2000 after 80 hrs. (Short solutions have the graph.)

10. a. For convenience, let 1941 correspond to \( t = 0 \) and define \( M(t) \) to be the population of India. The Malthusian growth model is

\[
\frac{dM}{dt} = kM, \quad M(0) = 319 \text{ (million)}. 
\]

The solution to this is \( M(t) = 319e^{kt} \). Since the population in 1961 \( (t = 20) \) is 439 million, the Malthusian growth model gives \( M(20) = 319e^{20k} = 439 \), so \( k = \frac{1}{20} \ln \left( \frac{439}{319} \right) \approx 0.015965 \). The population from this model for 1951 is given by

\[
P(10) = 319e^{0.015965(10)} = 374.2 \text{ million}.
\]

The percent error is \( 100 \times \left| \frac{374.2 - 361}{361} \right| = 3.7\% \).

b. We solve the separable differential equation

\[
\frac{dP}{dt} = (at + b)p,
\]

\[
\int \frac{dP}{P} = \int (at + b)dt,
\]

\[
\ln |P| = \frac{at^2}{2} + bt + C,
\]

\[
P(t) = e^{\exp \left( \frac{at^2}{2} + bt + C \right)} = A \cdot e^{\exp \left( \frac{at^2}{2} + bt \right)}. 
\]

From the initial data, \( P(0) = 319 \), it follows that \( A = 319 \). From the population data in 1951 and 1961, we have

\[
P(10) = 361 = 319e^{50a+10b} \quad \text{and} \quad P(20) = 439 = 319e^{200a+20b}.
\]

Thus,

\[
e^{50a+10b} = \frac{361}{319} = 1.13166 \quad \text{and} \quad e^{200a+20b} = \frac{439}{319} = 1.3762.
\]

Taking logarithms gives

\[
50a + 10b = \ln(1.13166) = 0.123687 \quad \text{and} \quad 200a + 20b = \ln(1.3762) = 0.319308.
\]

Multiply the first equation by -2 and add it to the second equation to get \( 100a = 0.319308 - 0.247373 \) or \( a = 0.00071935 \). Similarly, multiply the first equation by -4 and add it to the second equation to obtain \( -20b = 0.319308 - 0.494747 \) or \( b = 0.0087720 \). Thus, the nonautonomous Malthusian growth model becomes

\[
P(t) = 319e^{0.00035967t^2+0.0087720t}.
\]

c. The Malthusian growth model gives the population in 1991 as \( M(50) = 708.7 \) million, while the nonautonomous Malthusian growth model gives \( P(50) = 1,215.6 \) million. The percent error from the actual population of 846 million for the Malthusian growth model is 16.2%, while the percent error for the nonautonomous Malthusian growth model is 43.7%. So in this case, the Malthusian growth model is the better model. See the short solutions for the graph.