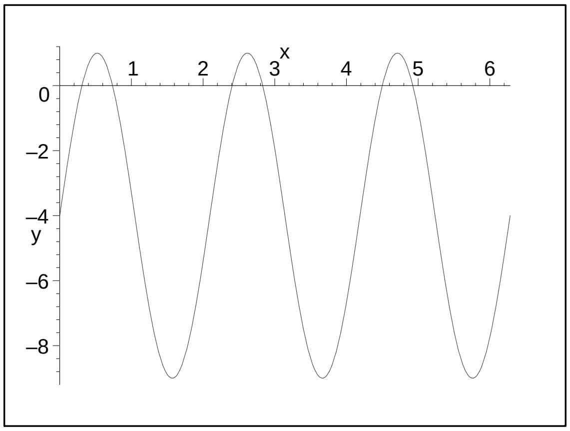
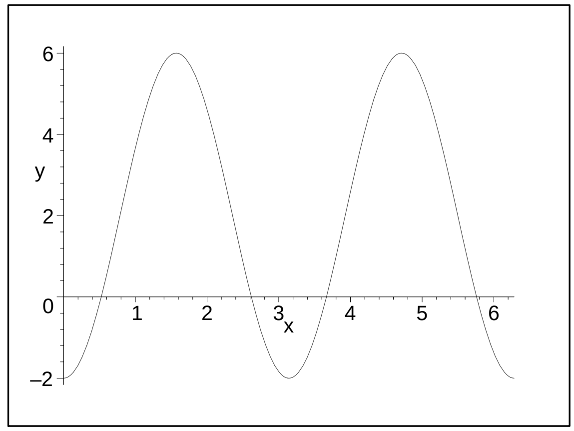
1. The function, $y = 5\sin(3x) - 4$, has a period of $x = 2\pi/3$. The function oscillates about $y = -4$, the vertical shift, with an amplitude of 5. It begins at $(0, -4)$, goes to a maximum at $(\pi/6, 1)$, continues through $(\pi/3, -4)$, then reaches a minimum at $(\pi/2, -9)$, and ends its cycle at $(2\pi/3, -4)$. The maxima occur at $x = \pi/6, 5\pi/6, 3\pi/2$. The graph of the function is below.



Problem 1



Problem 2

2. a. The function, $y = 2 - 4\cos(2x)$, has a period of $x = \pi$. The function oscillates about $y = 2$ with an amplitude of 4. It begins at a minimum at $(0, -2)$, goes to a maximum at $(\pi/2, 6)$, then ends its cycle at $(\pi, -2)$. The maxima occur at $x = \pi/2, 3\pi/2$ with y values of 6. The graph of the function is above.

b. The equivalent form has $A = 2$, $B = 4$, and $\omega = 2$. Since the amplitude has a negative sign, we phase shift the function by half a period or $\phi = \frac{\pi}{2}$. Thus,

$$y(x) = 2 + 4\cos\left(2\left(x - \tfrac{\pi}{2}\right)\right).$$

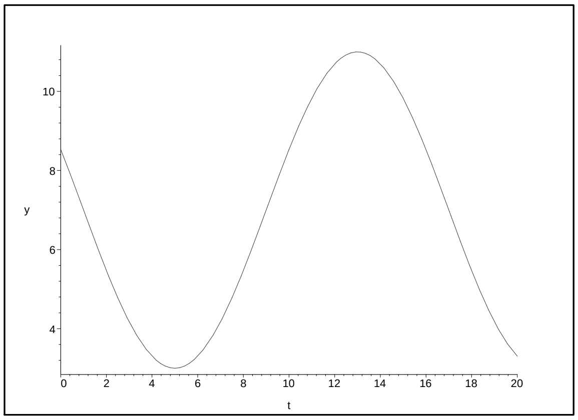
c. The notes show that the equivalent sine model is shifted a quarter period from the cosine model in Part b. Thus, $C = A$, $D = B$, $\nu = \omega$, and $\psi = \phi - \frac{\pi}{4}$ or

$$y(x) = 2 + 4\sin\left(2\left(x - \tfrac{\pi}{4}\right)\right).$$

3. a. The function, $y(t) = 7 - 4\cos\left(\frac{\pi}{8}(t - 5)\right)$, has a period of $T = 16$. The function oscillates about $y = 7$ (vertical shift) with an amplitude of 4. The phase shift is $\phi = 5$. There is an absolute maximum at $(t_{max}, y(t_{max})) = (13, 11)$. There is an absolute minimum at $(t_{min}, y(t_{min})) = (5, 3)$. The graph of the function is below.

b. The equivalent form has $A = 7$, $B = 4$, and $\omega = \frac{\pi}{8}$. Since the amplitude has a negative sign, we phase shift the function by half a period. It follows that $\phi = 5 + 8 = 13$. Thus,

$$y(t) = 7 + 4\cos\left(\tfrac{\pi}{8}(t - 13)\right).$$



Problem 3

c. The notes show that the equivalent sine model is shifted a quarter period from the cosine model in Part b. Thus, $C = A$, $D = B$, $\nu = \omega$, and $\psi = \phi - 4$ or

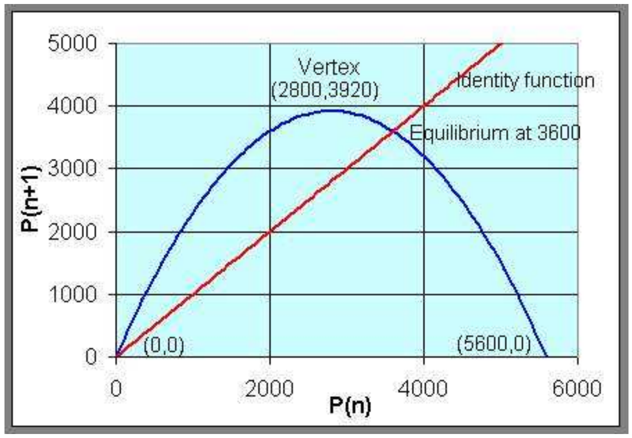
$$y(t) = 7 + 4\sin\left(\tfrac{\pi}{8}(t - 9)\right).$$

4. a. For the Logistic growth model, $P_{n+1} = F(P_n) = 2.8P_n - 0.0005P_n^2$ with $P_0 = 1000$, then

$$\begin{aligned} P_1 &= 2.8(1000) - 0.0005(1000)^2 = 2800 - 500 = 2300 \\ P_2 &= 2.8(2300) - 0.0005(2300)^2 = 3795 \end{aligned}$$

b. At equilibrium, $P_e = P_n = P_{n+1}$, so $P_e = 2.8P_e - 0.0005P_e^2$ or $P_e(1.8 - 0.0005P_e) = 0$. One solution is $P_e = 0$, and the other equilibrium satisfies $1.8 - 0.0005P_e = 0$ or $P_e = \frac{1.8}{0.0005} = 3600$. The derivative of the updating function is $F'(P) = 2.8 - 0.001P$. At $P_e = 0$, $F'(0) = 2.8 > 1$, so this equilibrium is unstable with solutions monotonically growing away from $P_e = 0$. At $P_e = 3600$, $F'(3600) = 2.8 - 3.6 = -0.8 > -1$, so the higher equilibrium is stable with solutions oscillating, but approaching $P_e = 3600$.

c. We see that $F(P) = P(2.8 - 0.0005P)$, so the updating function has P-intercepts at $P = 0$ and $P = 5600$. The vertex has $P_v = 2800$, so $F(2800) = 3920$, which gives the vertex $(2800, 3920)$. The updating function intersects the identity function at the equilibria, $(0,0)$ and $(3600, 3600)$. The graph is shown below.



5. a. For the population model with the Allee effect, $N_{n+1} = N_n + 0.1N_n\left(1 - \frac{1}{9}(N_n - 5)^2\right)$ with (population in thousands) $N_0 = 4$, the next two generations are

$$\begin{aligned} N_1 &= 4 + 0.1(4)\left(1 - \frac{1}{9}(4-5)^2\right) = 4.356 \\ N_2 &= 4.356 + 0.1(4.356)\left(1 - \frac{1}{9}(4.356-5)^2\right) = 4.771 \end{aligned}$$

in thousands of birds.

b. $N_e = N_e + 0.1N_e\left(1 - \frac{1}{9}(N_e - 5)^2\right)$, so $0.1N_e\left(1 - \frac{1}{9}(N_e - 5)^2\right) = 0$. Thus, $N_e = 0$ or $(N_e - 5)^2 = 9$. It follows that the equilibria are $N_e = 0$, 2, and 8.

c. From the expanded model, $N_{n+1} = A(N_n) = \frac{37}{45}N_n + \frac{1}{9}N_n^2 - \frac{1}{90}N_n^3$, the derivative is $A'(N) = \frac{37}{45} + \frac{2}{9}N - \frac{1}{30}N^2$. At $N_e = 0$, $A'(0) = \frac{37}{45}$, so this equilibrium is a stable equilibrium with solutions monotonically approaching 0. At $N_e = 2$, $A'(2) = \frac{17}{15}$, so this equilibrium is an unstable equilibrium with solutions monotonically moving away from 2. At $N_e = 8$, $A'(8) = \frac{7}{15}$, so this equilibrium is a stable equilibrium with solutions monotonically approaching 8.

d. Biologically, these results imply that if the population is below 2 thousand, then it will go to extinction ($N_e = 0$). If the population is above 2 thousand, then the population of birds will grow to a carrying capacity of $N_e = 8$ thousand.

6. The volume of the open box satisfies the **Objective function**

$$V(x, y) = x^2y.$$

The **Constraint condition** on the surface area of this box is given by

$$SA = x^2 + 4xy = 600.$$

This constraint condition yields $y = \frac{600-x^2}{4x}$, which when substituted into the objective function produces a function of one variable:

$$V(x) = x^2\left(\frac{600 - x^2}{4x}\right) = \frac{1}{4}(600x - x^3).$$

Differentiating this quantity, we obtain

$$\frac{dV}{dx} = \frac{1}{4}(600 - 3x^2),$$

which when set equal to zero gives $x = 10\sqrt{2}$. (Take only the positive root.) This value of x gives the optimal length of one side of the base, which when substituted into the formula above gives $y = 5\sqrt{2}$. It follows that the maximum volume for this box is $V(x) = 1000\sqrt{2}$.

7. Combining the number of drops with the energy function, we have

$$E(h) = hN(h) = h\left(1 + \frac{10}{h-1}\right) = h\left(\frac{h-1+10}{h-1}\right) = \frac{h^2+9h}{h-1}.$$

This is differentiated to give

$$E'(h) = \frac{(h-1)(2h+9) - (h^2+9h)}{(h-1)^2} = \frac{h^2-2h-9}{(h-1)^2}.$$

A minimum occurs when $h^2 - 2h - 9 = 0$, so

$$h = 1 \pm \sqrt{10} = -2.1623, 4.1623.$$

It follows that the minimum energy occurs when $h = 1 + \sqrt{10} = 4.1623$ m, which give the height that a crow should fly to minimize the energy needed to break open a walnut.

8. The area of the brochure is $A = xy = 125$, where x is the width of the page and y is the length of the page. The area of the printed page, which is to be maximized is given by

$$P = (x-4)(y-5).$$

From the constraint on the page area, we have $y = 125/x$, which when substituted above gives

$$P(x) = (x-4)\left(\frac{125}{x} - 5\right) = 125 - \frac{500}{x} - 5x + 20 = 145 - 500x^{-1} - 5x.$$

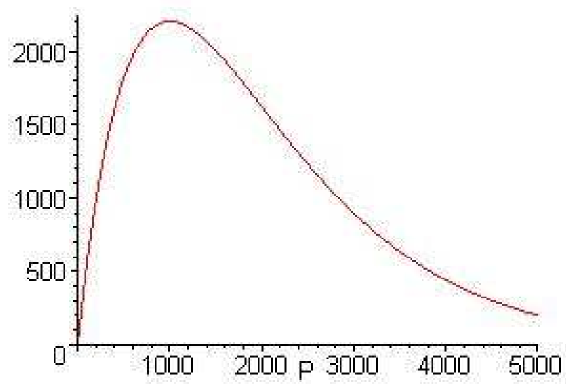
The maximum is found by differentiation, which gives

$$P'(x) = 500x^{-2} - 5 = \frac{5(100-x^2)}{x^2}.$$

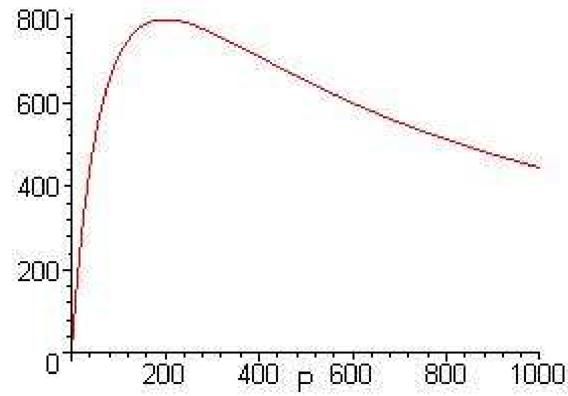
This is zero when $x = 10$. It follows that $y = 12.5$. So the brochure has the dimensions 10×12.5 with the printed region having dimensions 6×7.5 or 45 in^2.

9. a. If $P_0 = 100$, then $P_1 = 600e^{-0.1} = 542.9$ and $P_2 = 6(542.9)e^{-0.5429} = 1892.75$.

b. The derivative is $R'(P) = 6e^{-0.001P}(1 - 0.001P)$. The critical P_c occurs at $P_c = 1000$, so there is a maximum at $(1000, 6000e^{-1}) = (1000, 2207)$. The graph passes through the origin, so $(0,0)$ is the only intercept. Since $\lim_{P\to\infty} R(P) = 0$, the is a horizontal asymptote at $R = 0$. The second derivative is $R''(P) = -0.006e^{-0.001P}(2 - 0.001P)$, which is zero at $P = 2000$. Thus, there is a point of inflection at $(2000, 12000e^{-2}) = (2000, 1624)$. The graph is below.



Problem 9



Problem 10

c. The equilibria satisfy $P_e = 6P_e e^{-0.001P_e}$, so either $P_e = 0$ or $1 = 6e^{-0.001P_e}$. The latter gives $P_e = 1000\ln(6) \simeq 1792$. For $P_e = 0$, $R'(0) = 6 > 1$, so this equilibrium is unstable with solutions moving monotonically away. For $P_e = 1792$, $R'(1792) = -0.7918$, so this equilibrium is stable with solutions oscillating and moving toward the equilibrium.

10. a. If $P_0 = 500$, then $P_1 = 8000/3.5^2 = 653.06$ and $P_2 = 574.346$.

b. The derivative is $H'(P) = \dfrac{16(1+0.005P)^2 - 32P(1+0.005P)(0.005)}{(1+0.005P)^4} = \dfrac{16(1-0.005P)}{(1+0.005P)^3}$. The critical P_c occurs at $P_c = 200$, so there is a maximum at $(200, 800)$. The graph passes through the origin, so $(0,0)$ is the only intercept. Since $\lim_{P\to\infty} H(P) = 0$, the is a horizontal asymptote at $H = 0$. The second derivative is
$H''(P) = \dfrac{-0.08(1+0.005P)^3 - 0.24(1-0.005P)(1+0.005P)^2}{(1+0.005P)^6} = \dfrac{-0.16(2-0.005P)}{(1+0.005P)^4}$,
which is zero at $P = 400$. Thus, there is a point of inflection at $(400, 6400/9) = (400, 711)$. The graph is above.

c. The equilibria satisfy $P_e = 16P_e/(1+0.005P_e)^2$, so either $P_e = 0$ or $(1+0.005P_e)^2 = 16$. The latter gives $P_e = 600$ (neglecting the negative solution). For $P_e = 0$, $H'(0) = 16 > 1$, so this equilibrium is unstable with solutions moving monotonically away. For $P_e = 600$, $H'(600) = -0.5$, so this equilibrium is stable with solutions oscillating and moving toward the equilibrium.

11. a. The time as a function of x is given by

$$T(x) = \frac{50-x}{15} + \frac{(x^2+1600)^{1/2}}{9}.$$

b. We differentiate $T(x)$ to find the minimum time,

$$T'(x) = -\frac{1}{15} + \frac{1}{9}\left(\frac{1}{2}(x^2+1600)^{-1/2}2x\right) = -\frac{1}{15} + \frac{x}{9(x^2+1600)^{1/2}}.$$

Setting this derivative equal to zero gives

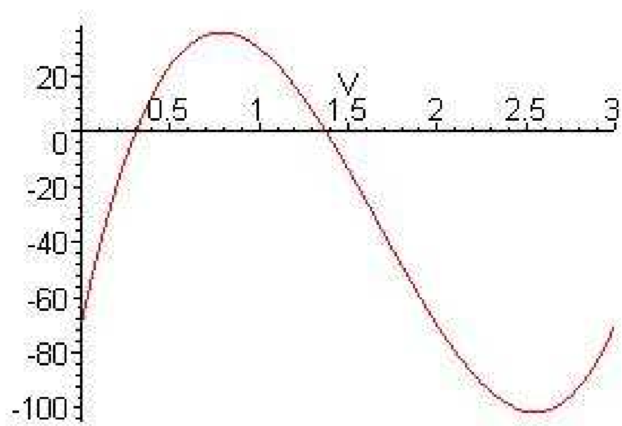
$$\frac{x}{9(x^2+1600)^{1/2}} = \frac{1}{15}$$

$$
\begin{aligned}
5x &= 3(x^2+1600)^{1/2} \\
25x^2 &= 9(x^2+1600) \\
16x^2 &= 14400 \\
x^2 &= 900
\end{aligned}
$$

This implies $x = 30$ m produces the minimum time. $T(30) = \frac{20}{15} + \frac{50}{9} = \frac{62}{9} = 6.89$ sec. We check the endpoints $T(0) = \frac{70}{9} = 7.778$ sec and $T(50) = \frac{10\sqrt{41}}{9} = 7.11$ sec, confirming the optimal escape strategy is for the rabbit to run 20 m along the road, then run straight toward the burrow.

12. a. At rest, $V(t) = -70 = 50t(t-2)(t-3) - 70$, so $50t(t-2)(t-3) = 0$. Thus, the membrane is at rest when $t = 0$, 2, and 3.

b. To find the extrema, we first write $V(t) = 50(t^3 - 5t^2 + 6t) - 70$, then the derivative is $V\,'(t) = 50(3t^2 - 10t + 6)$. By the quadratic formula, $t = \frac{5}{3} \pm \frac{\sqrt{7}}{3} = 0.7847, 2.5486$. Substituting these values into the membrane equation gives the peak of the action potential at $t = 0.7847$ with a membrane potential of $V(0.7847) = 35.63$ mV, while the minimum potential (most hyperpolarized state) occurs at $t = 2.5486$ with a membrane potential of $V(2.5486) = -101.56$ mV. Below is a graph for this model of membrane potential.



13. The **objective function** is given by:

$$S(x,y) = 2x^2 + 7xy.$$

The constraint condition is given by:

$$V = x^2y = 50,000 \text{ cm}^3, \quad \text{so,} \quad y = \frac{50,000}{x^2}.$$

Thus,

$$S(x) = 2x^2 + \frac{350,000}{x}.$$

Differentiating we have,

$$S'(x) = 4x - \frac{350,000}{x^2}.$$

Solving $S'(x) = 0$, so $x^3 = \frac{350{,}000}{4} = 87{,}500$ or $x = 44.395$. It follows $y = 25.37$. Thus, the minimum amount of material needed is $S(44.395) = 11{,}825.6$ cm^2.

14. a. $L(0) = 0.24$ m (24 cm) is the birth size a leopard shark (L-intercept). For large t, $L(t) \to 1.6$ m. The graph of this von Bertalanffy equation is shown below. Sexual maturity is found by solving $L(t) = 0.5 = 1.6(1 - 0.85e{-}0.08t)$ or $1.36e^{-0.08t} = 1.1$ or $e^{0.08t} = 1.236$. It follows that sexual maturity occurs at $t = 2.652$ yr.

b. The composite function is given by

$$W(t) = 4.5(1.6(1 - 0.85e^{-0.08t}))^3 = 18.432(1 - 0.85e^{-0.08t})^3.$$

The intercept is $W(0) = 0.0622$ kg, while for large t, $W(t) \to 18.432$ kg. The graph of this function is shown below.

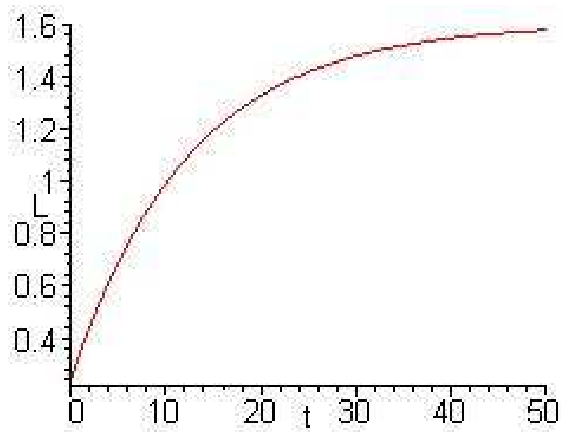
c. By the chain rule, the derivative of $W(t)$ is

$$W'(t) = 3(18.432)(1 - 0.85e^{-0.08t})^2(-0.85)(-0.08)e^{-0.08t} = 3.76e^{-0.08t}(1 - 0.85e^{-0.08t})^2.$$

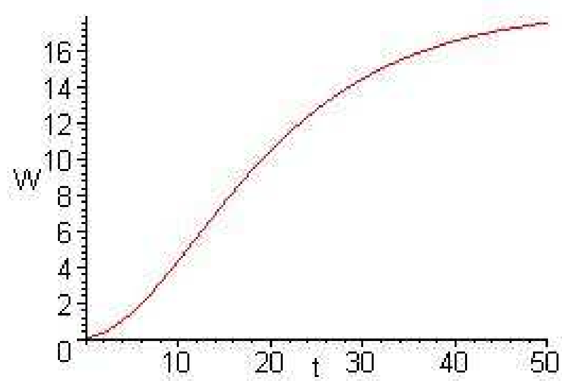
By the product rule and chain rule, the second derivative is

$$\begin{aligned} W''(t) &= 3.76\left(2e^{-0.08t}(1 - 0.85e^{-0.08t})(-0.85)(-0.08)e^{-0.08t} - 0.08e^{-0.08t}(1 - 0.85e^{-0.08t})^2\right) \\ W''(t) &= 3.76e^{-0.08t}(1 - 0.85e^{-0.08t})(0.204e^{-0.08t} - 0.08) \end{aligned}$$

$W''(t) = 0$ when either $1 - 0.85e^{-0.08t} = 0$ or $0.204e^{-0.08t} - 0.08 = 0$. The first is zero when $t = -2.03$ yr, while the second is zero when $t = 11.7$ yr. It follows that the maximum weight gain occurs at age $t = 11.7$ yr with a weight gain of $W'(11.7) = 0.655$ kg/yr.



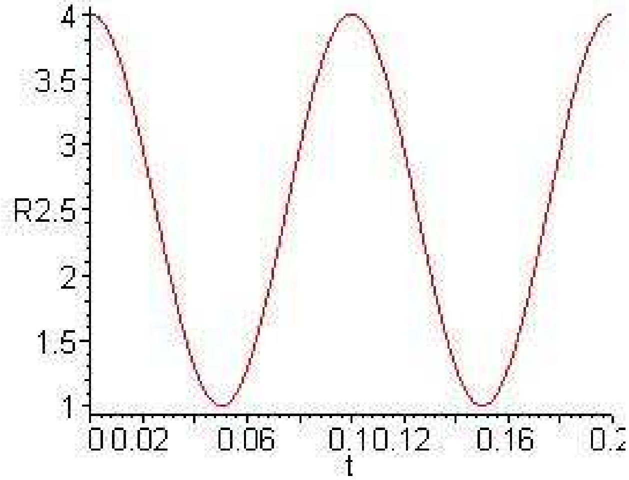
Problem 13a



Problem 13b

15. a. The periodic contractions of 10/min implies that the period is 0.1 min. Thus, $0.1\omega = 2\pi$ or $\omega = 20\pi$. The average value $A = \frac{4+1}{2} = 2.5$, while the amplitude is given by $B = 4 - 2.5 = 1.5$. Thus, the radius of the small intestine is given by

$$R(t) = 2.5 + 1.5\cos(20\pi t).$$



b. The graph of $R(t)$ for $t \in [0, 0.2]$ is shown below. The maxima occur at $t = 0, 0.1, 0.2$ min, and the minima are halfway between the maxima with $t = 0.05, 0.15$ min.

c. The equivalent sine form of the model is phase shifted by a quarter period or $\frac{1}{40}$ min, so $\phi = 0 - \frac{1}{40}$. However, this is negative, so the principle phase shift requires adding one period or $\phi = -\frac{1}{40} + \frac{1}{10} = \frac{3}{40}$. The equivalent sine model is written:

$$R(t) = 2.5 + 1.5\sin\left(20\pi\left(t - \tfrac{3}{40}\right)\right).$$

16. a. The period is 365 days, so $365\omega = 2\pi$ or $\omega = \frac{2\pi}{365} \simeq 0.01721$. The average length of time is $\alpha = \frac{1162+327}{2} = 744.5$ min. The amplitude is given by $\beta = 1162 - 744.5 = 417.5$ min. The maximum occurs on day 170, so $\omega(170 - \phi) = \pi/2$ (based on the maximum of the sine function). Thus, $170 - \phi = \frac{365}{4} = 91.25$ or $\phi = 78.75$ day. It follows that

$$L(t) = 744.5 + 417.5\sin(0.01721(t - 78.75)).$$

The length of day for Ground Hog's day is $L(32) = 744.5 + 417.5\sin(0.01721(32 - 78.75)) = 443.7$ min in Anchorage.

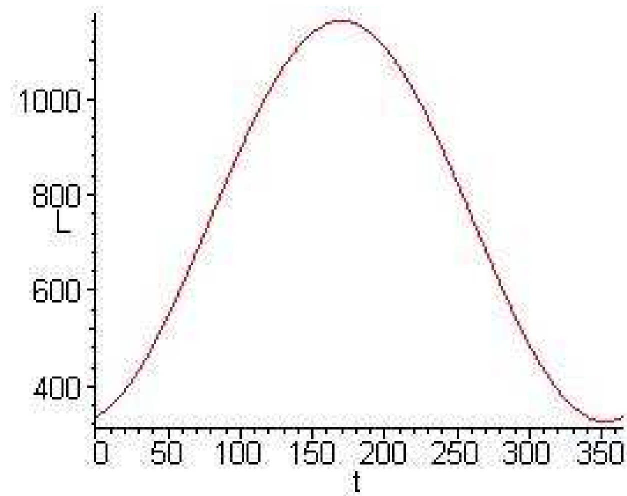
b. The equivalent cosine form of the model is phase shifted by a quarter period or 91.25 days, so $\phi = 78.75 + 91.25 = 170$. Also, one can use that the maximum of the cosine model occurs at the phase shift. Thus, the equivalent cosine model is written:

$$L(t) = 744.5 + 417.5\cos(0.01721(t - 170)).$$

17. a. From P_3, we have $P_3 = 68.34 = 28.49(1+r)^3$, so $(1+r) = (68.34/28.49)^{1/3} = 1.33863$. Thus, $r = 0.33863$. Doubling time satisfies $2P_0 = P_0(1+r)^n$ or $n = \ln(2)/\ln(1+r) = 2.377$ decades or 23.77 years.

b. The model predicts the population in 2000 is $P_5 = 28.49(1.33863)^5 = 122.46$ million. The percent error is $100\frac{(122.46-99.93)}{99.93} = 22.55\%$.

c. From the logistic model, we obtain $P_1 = 39.32$ million and $P_2 = 52.79$ million.



d. To find equilibria, we solve $P_e = 1.48P_e - 0.0035P_e^2$, which gives $P_e = 0$ or $P_e = 137.14$ million. The derivative of the updating function is $F'(P) = 1.48 - 0.007P$, so $F'(137.14) = 0.52$. It follows that this equilibrium is stable with solutions monotonically approaching this carrying capacity equilibrium.

18. a. From the high and low temperatures, A is the average, so $A = 18°$C. The amplitude B is the difference between the maximum and the average, so $B = 8°$C. The period is 24 hr, so $24\omega = 2\pi$ or $\omega = \frac{\pi}{12} \simeq 0.2618$. The maximum temperature occurs at 4 PM ($t = 16$), so

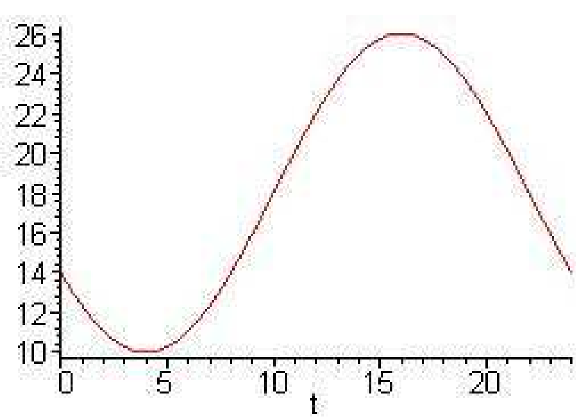
$$T(16) = 26 = 18 + 8\sin\left(\frac{\pi}{12}(16 - \phi)\right).$$

It follows that

$$\sin\left(\frac{\pi}{12}(16 - \phi)\right) = 1 \quad \text{or} \quad \frac{\pi}{12}(16 - \phi) = \frac{\pi}{2}.$$

Hence, $\phi = 10$. The sine model becomes

$$R(t) = 18 + 8\sin(0.2618(t - 10)).$$



b. The equivalent cosine form of the model is phase shifted by a quarter period or 6 hr, so $\phi = 10 + 6 = 16$. Again the phase shift for the cosine model is easy as it corresponds to the maximum. So, we obtain the equivalent cosine model:

$$R(t) = 18 + 8\cos(0.2618(t - 16)).$$