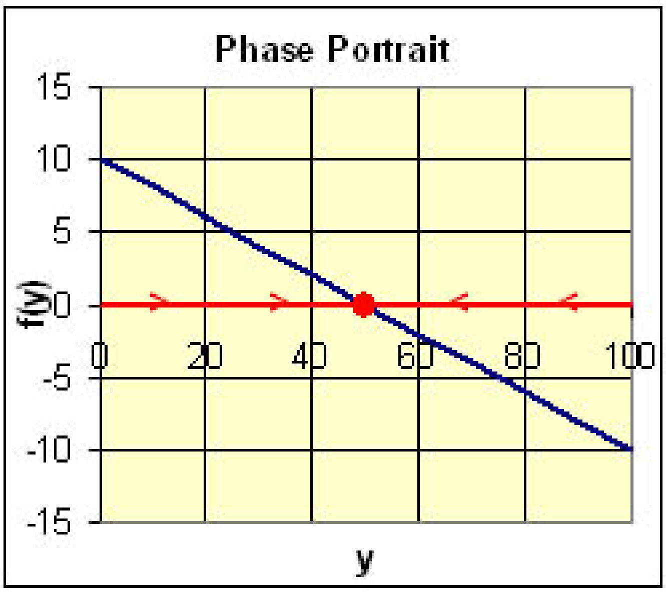
1. a. This differential equation can be written

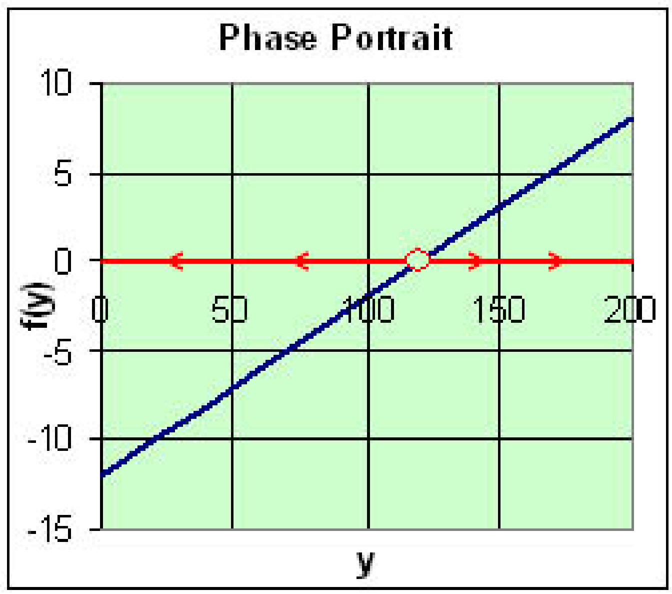
$$\frac{dy}{dt} = -0.2(y - 50).$$

Substitute $z(t) = y(t) - 50$, then $\frac{dz}{dt} = -0.2\,z$ with $z(0) = z_0$ (some initial value). Thus, $z(t) = z_0 e^{-0.2\,t} = y(t) - 50$. It follows that

$$y(t) = 50 + z_0 e^{-0.2\,t},$$

for some constant z_0. The equilibrium occurs at $y_e = 50$, and it's stable. Below to the left is a sketch of the right hand side of this differential equation and the phase portrait is on the y-axis.





b. This differential equation can be written

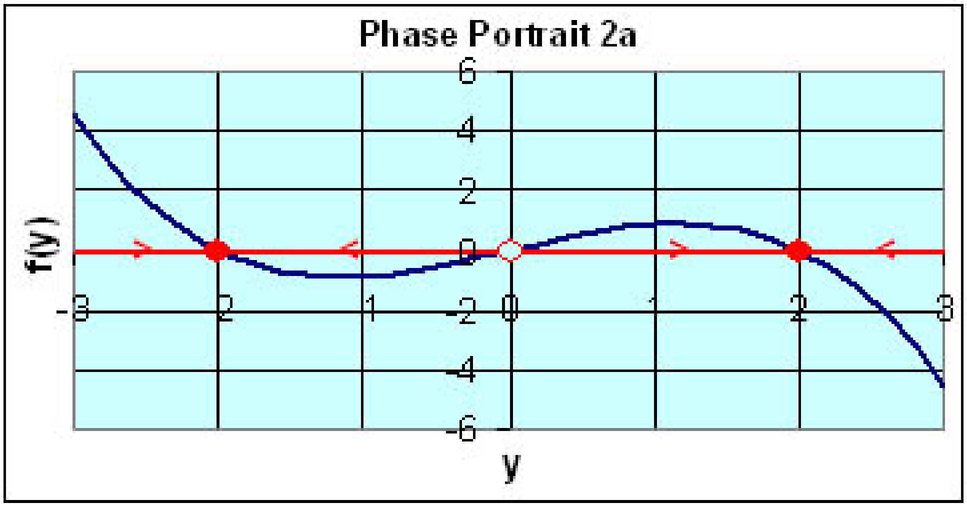
$$\frac{dy}{dt} = 0.1(y - 120).$$

Substitute $z(t) = y(t) - 120$, then $\frac{dz}{dt} = 0.1\,z$ with $z(0) = z_0$ (some initial value). Thus, $z(t) = z_0 e^{0.1\,t} = y(t) - 120$. It follows that

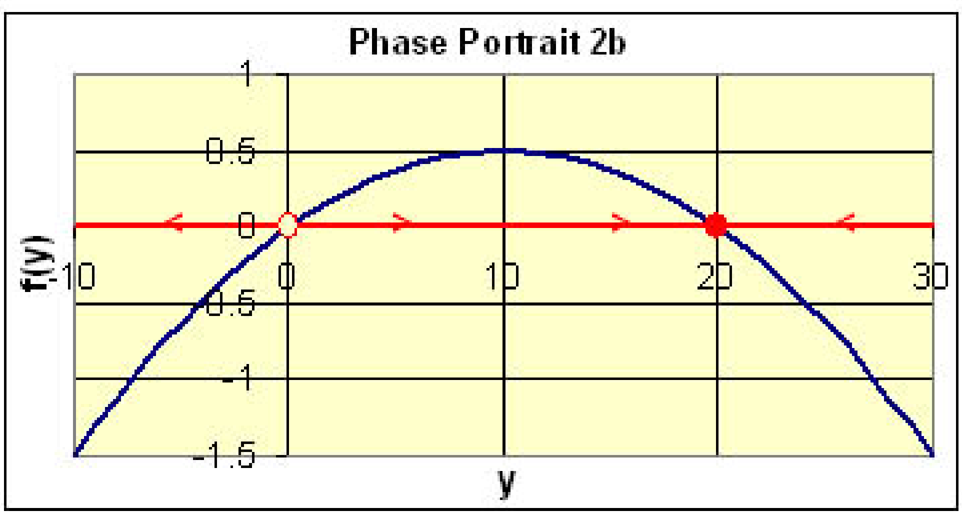
$$y(t) = 120 + z_0 e^{0.1\,t},$$

for some constant z_0. The equilibrium occurs at $y_e = 120$, and it's unstable. Above to the right is a sketch of the right hand side of this differential equation and the phase portrait is on the y-axis.

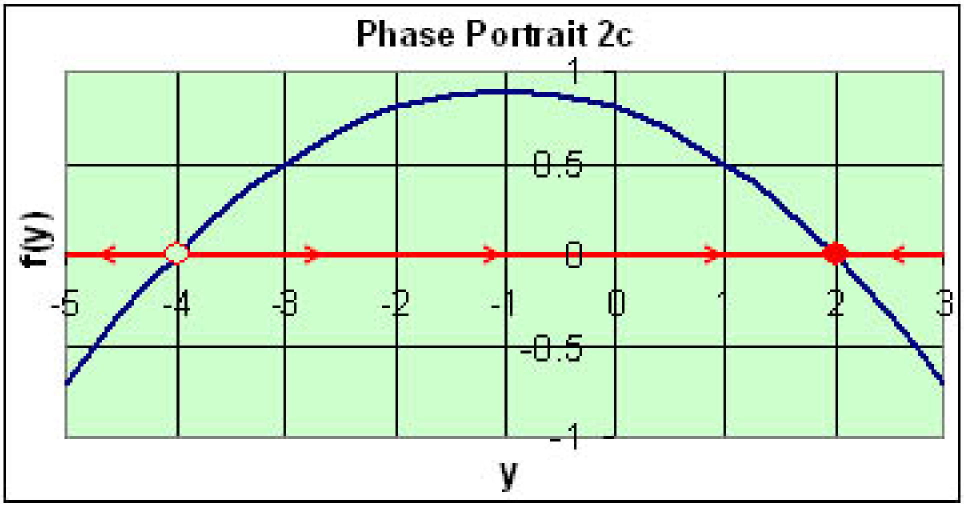
2. a. For the differential equation, $\frac{dy}{dt} = 0.3\,y(4 - y^2)$,, the equilibria are easily found by setting the right hand side of the equation equal to zero. There are three equilibria $y_e = 0, \pm 2$. Below is the graph and the phase portrait.



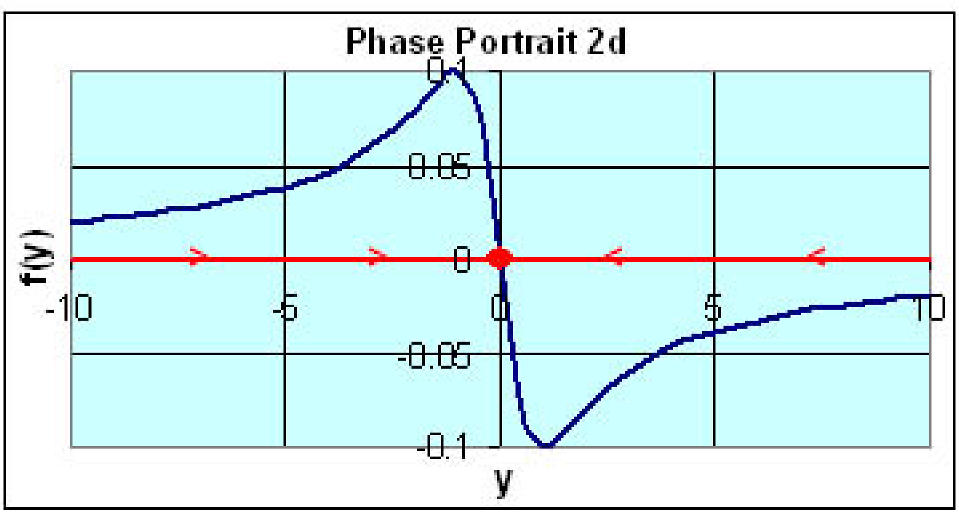
b. For the differential equation, $\frac{dy}{dt} = 0.1y\left(1 - \frac{y}{20}\right)$,, the equilibria are easily found by setting the right hand side of the equation equal to zero. There are two equilibria $y_e = 0, 20$. Below is the graph and the phase portrait.



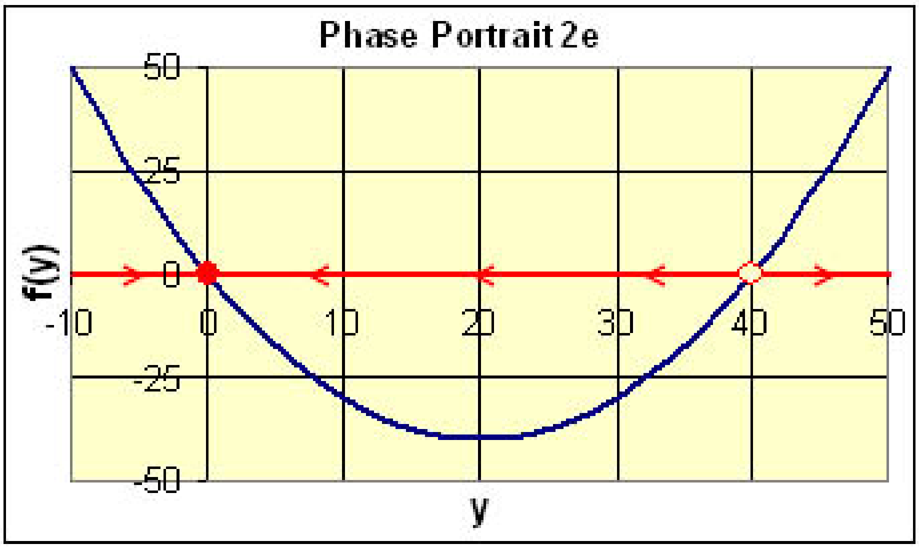
c. For the differential equation, $\dfrac{dy}{dt} = 0.8 - 0.2\,y - 0.1y^2$,, the equilibria are found by factoring the right hand side of the equation. There are two equilibria $y_e = -4, 2$. Below is the graph and the phase portrait.



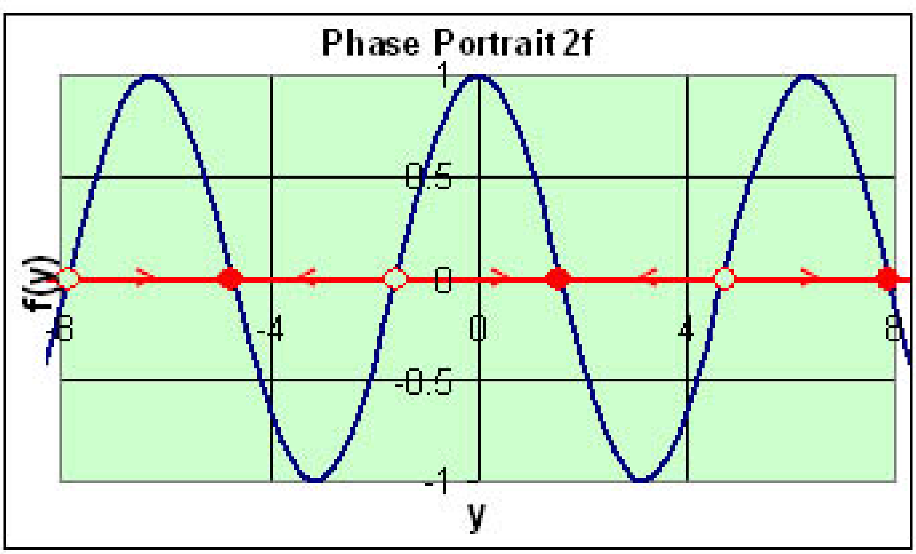
d. For the differential equation, $\dfrac{dy}{dt} = -\dfrac{0.2\,y}{1+y^2}$,, the equilibria are easily found by setting the right hand side of the equation equal to zero. There is one equilibrium $y_e = 0$. Below is the graph and the phase portrait.



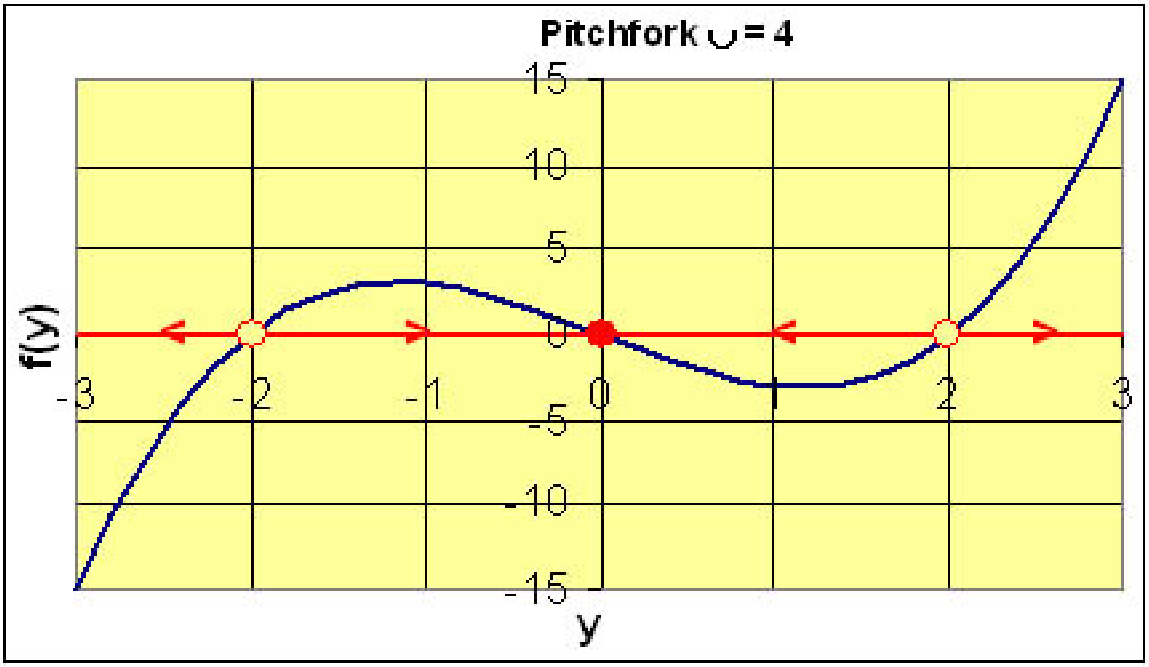
e. For the differential equation, $\dfrac{dy}{dt} = 0.1y^2 - 4\,y$,, the equilibria are found by factoring the right hand side of the equation. There are two equilibria $y_e = 0, 40$. Below is the graph and the phase portrait.

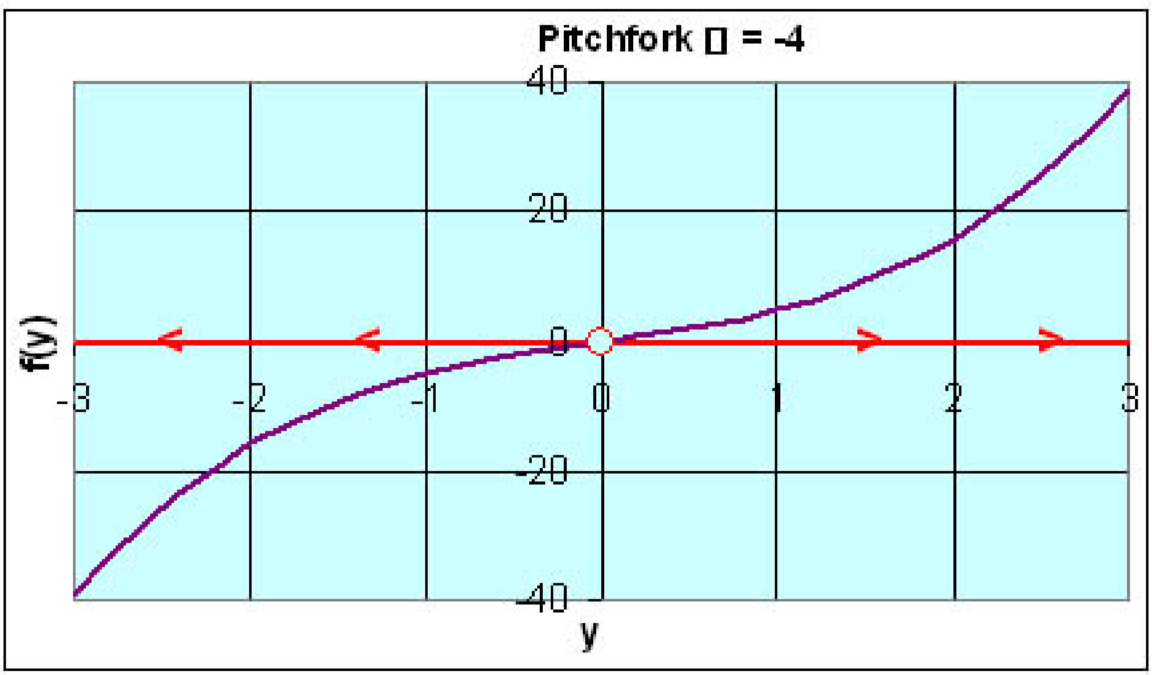


f. For the differential equation, $\dfrac{dy}{dt} = \cos(y)$,, the equilibria are found from the zeroes of the cosine function. There are infinity many equilibria $y_e = \frac{\pi}{2} + n\pi, n = 0, \pm 1, \pm 2, ...$ Below is the graph and the phase portrait.

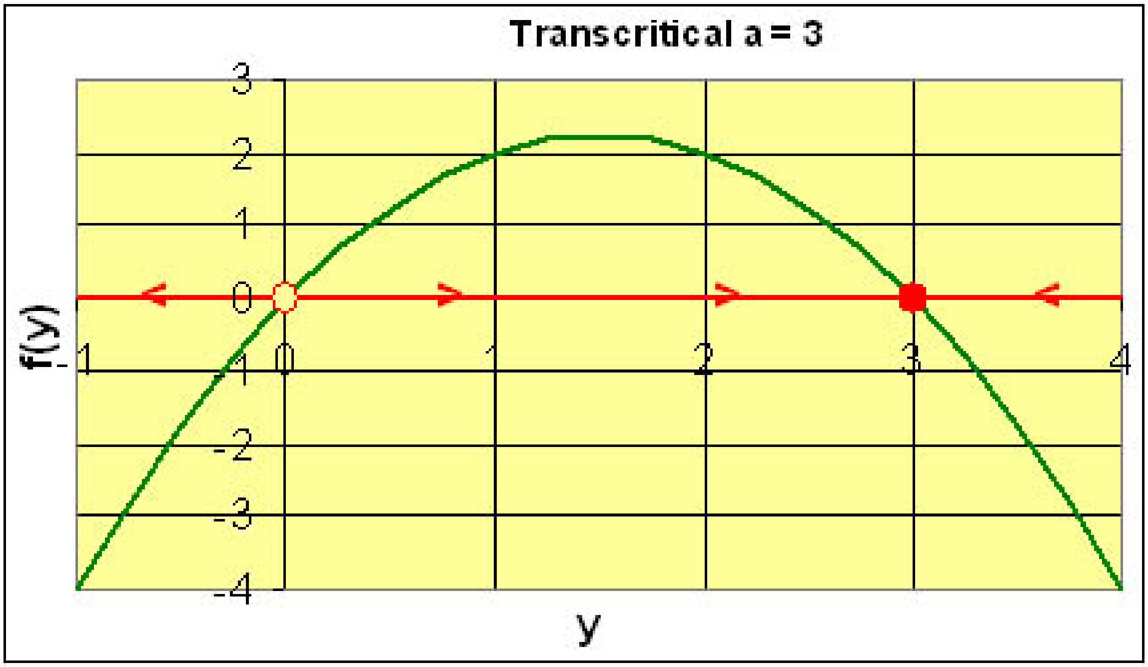


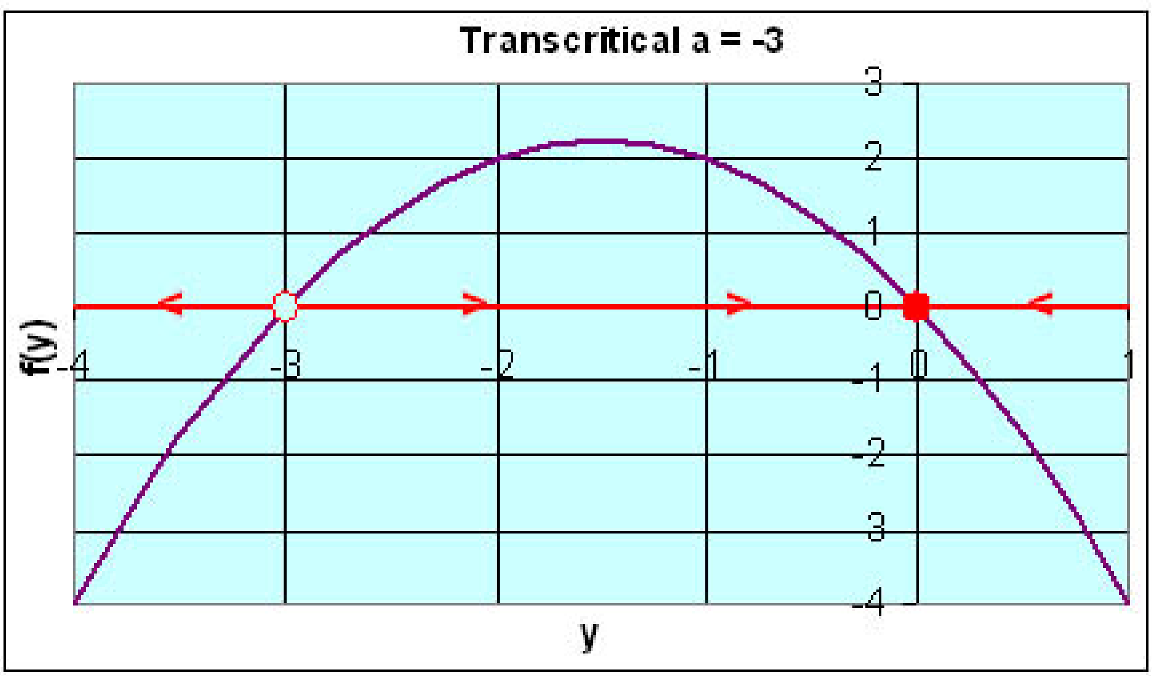
3. (Subcritical Pitchfork bifurcation) For the differential equation $\frac{dy}{dt} = y^3 - \alpha y$, the phase portraits are shown below. When $\alpha = 4$, there are three equilibria, $y_e = 0$, which is stable, and $y_e = \pm 2$, which are both unstable. When $\alpha = -4$, there is only one equilibrium, $y_e = 0$, which is unstable. The behavior switches from one equilibrium to three equilibria at $\alpha = 0$.



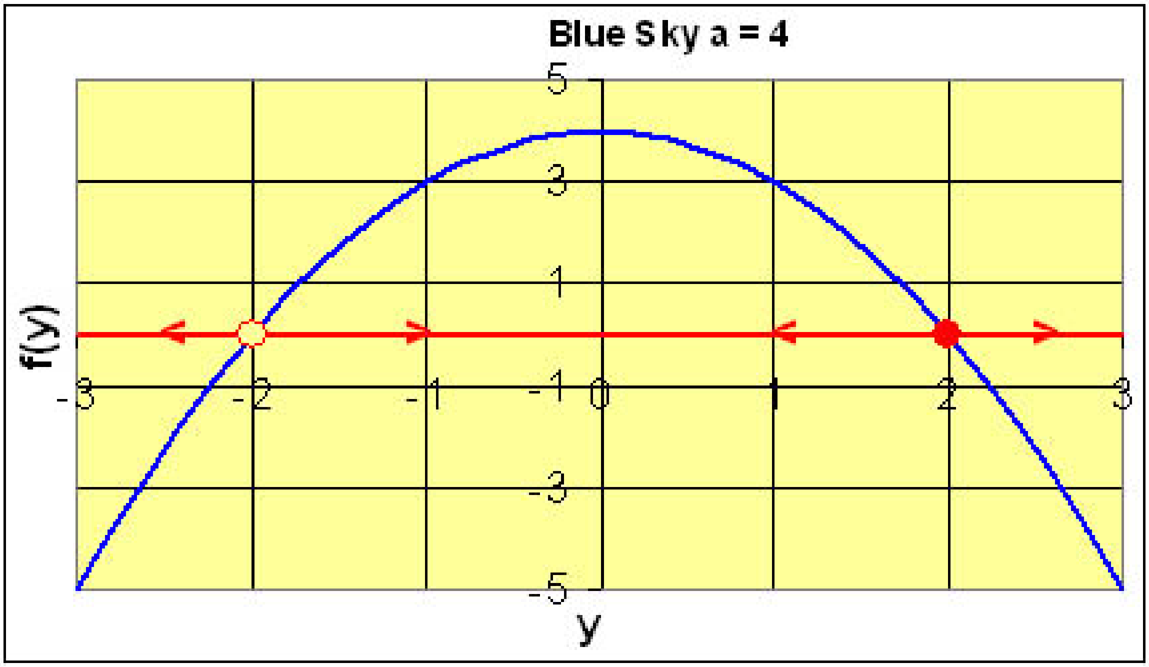


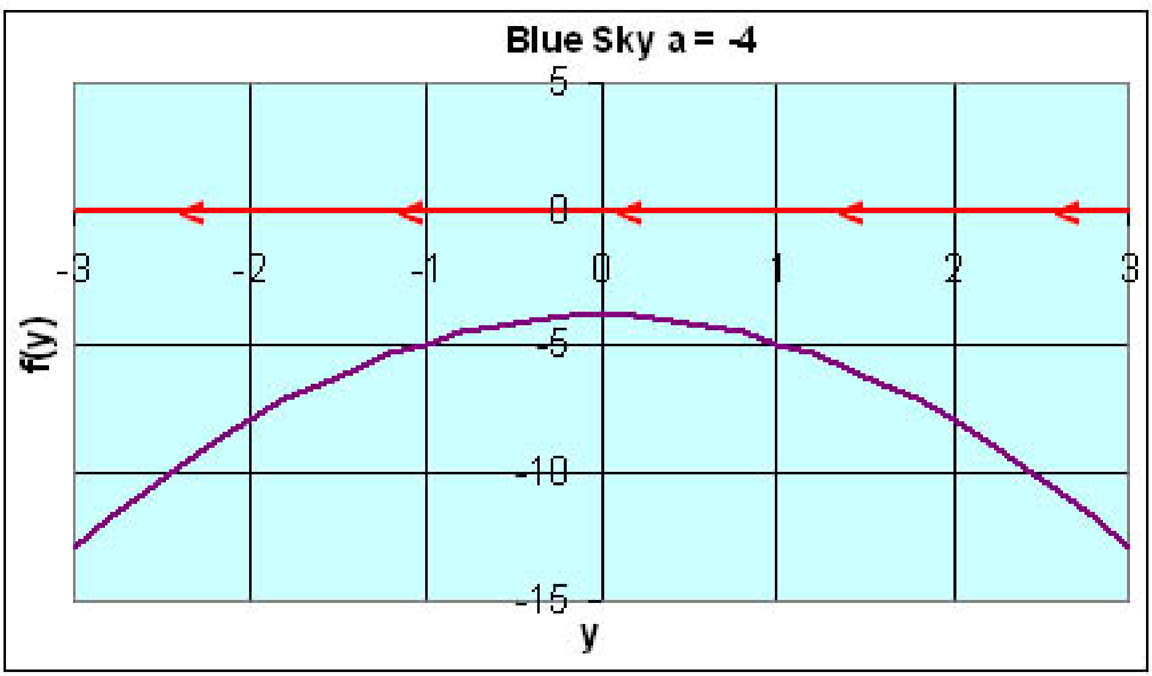
4. (Transcritical bifurcation) For the differential equation $\frac{dy}{dt} = \alpha y - y^2$, the phase portraits are shown below. When $\alpha = 3$, there are two equilibria, $y_e = 0$, which is unstable, and $y_e = 3$, which is stable. When $\alpha = -3$, there are two equilibria, $y_e = 0$, which is stable, and $y_e = -3$, which is unstable. At $\alpha = 0$, there is only one equilibrium ($y_e = 0$) and it switches stability at this value of α.



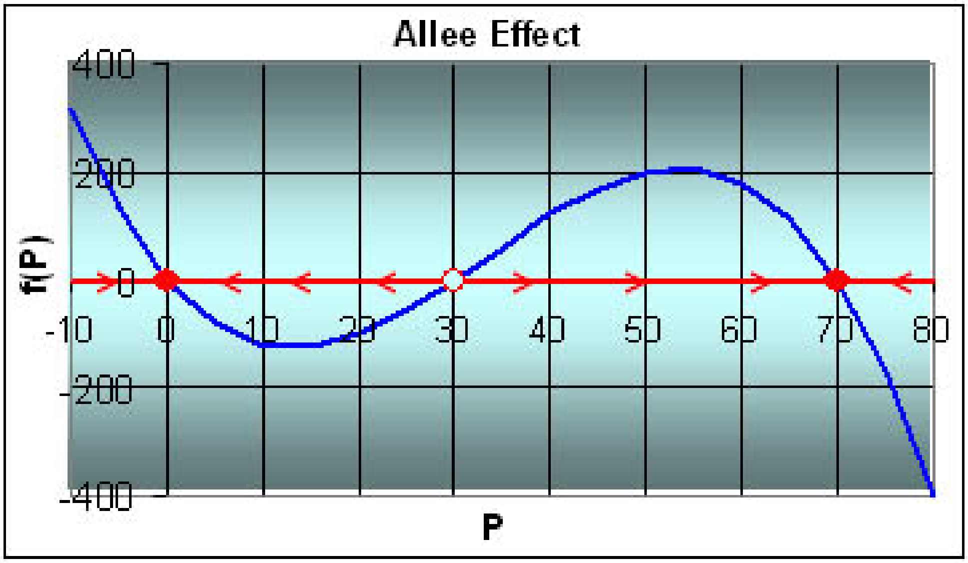


5. (Saddle-node or Blue sky bifurcation) For the differential equation $\frac{dy}{dt} = \alpha - y^2$, the phase portraits are shown below. When $\alpha = 4$, there are two equilibria, $y_e = -2$, which is unstable, and $y_e = 2$, which is stable. When $\alpha = -4$, there are no equilibria. At $\alpha = 0$, the behavior switches, and there is only one equilibrium ($y_e = 0$). In this case, $y_e = 0$ is said to be half-stable as it is stable to the right and unstable to the left.

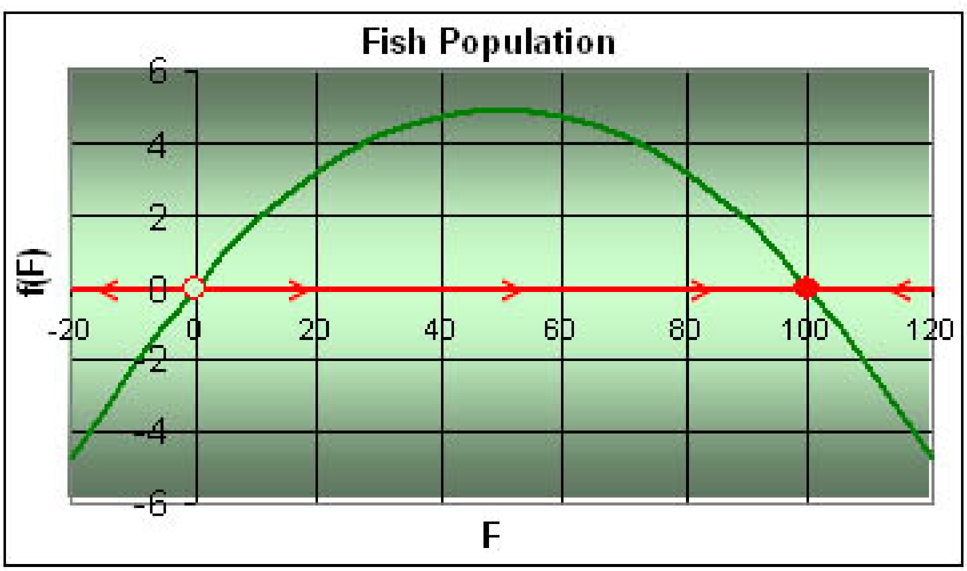




6. (Allee effect) For the model $\dfrac{dP}{dt} = P\left(4 - 0.01(P-50)^2\right)$, the phase portrait is shown below. There are three equilibria, $P_e = 0$, which is stable, $P_e = 30$, which is unstable, and $P_e = 70$, which is stable. The carrying capacity is 70, while the critical threshhold number of animals required to avoid extinction is 30.

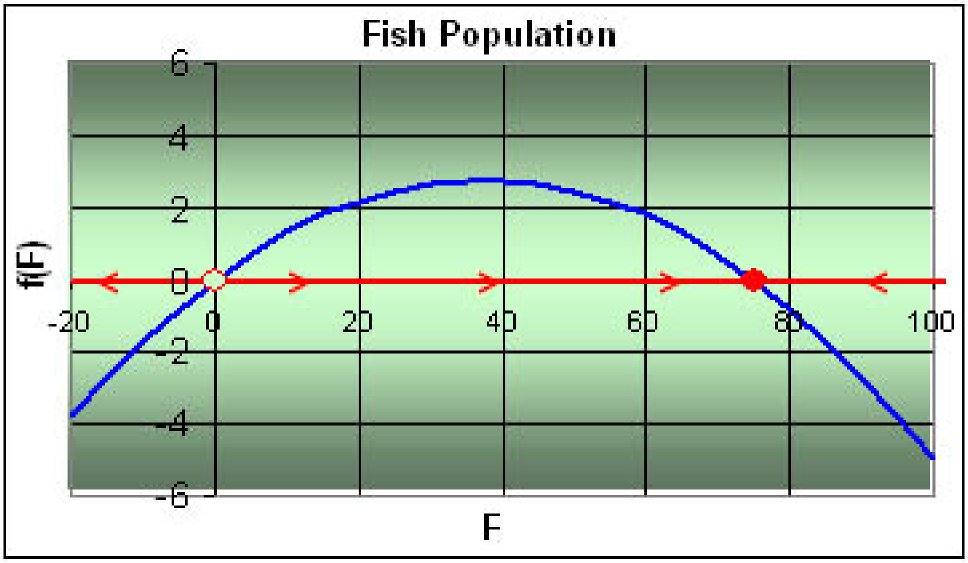


7. a. (Harvesting 1) For the model $\dfrac{dF}{dt} = 0.2\,F\left(1 - \dfrac{F}{100}\right) - hF$, the phase portrait is shown below. There are two equilibria, $F_e = 0$, which is unstable, and $F_e = 100$, which is stable and the carrying capacity.

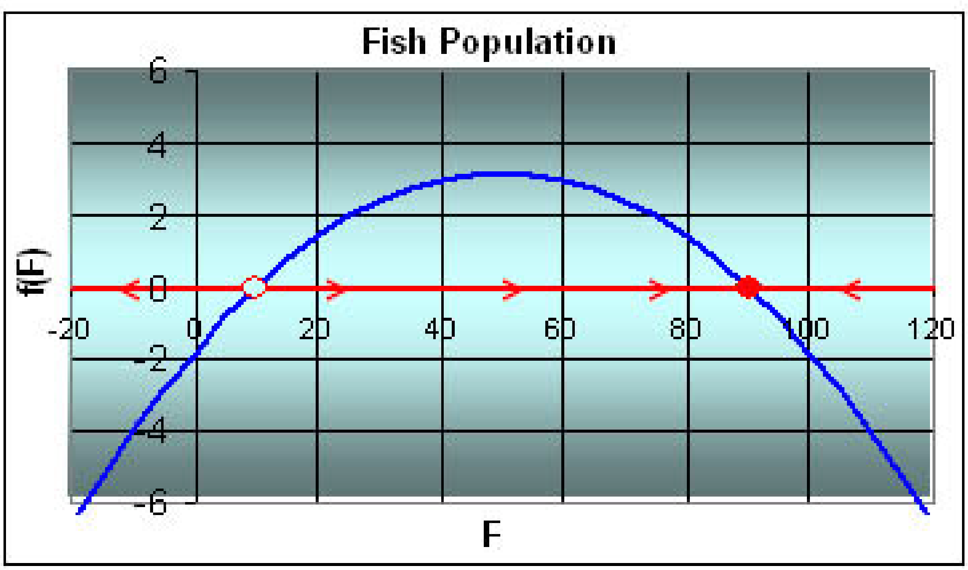


7 b. For $h = 0.05$, the phase portrait is shown below. There are two equilibria, $F_e = 0$, which is unstable, and $F_e = 75$, which is stable and the carrying capacity.

c. When $h = 0.2$, the only equilibrium is $F_e = 0$, and it is half stable. Thus, any level of fishing at or above $h = 0.2$ results in the fish going extinct.



8. a. (Harvesting 2) For the model $\frac{dF}{dt} = 0.2\,F\left(1 - \frac{F}{100}\right) - h$, with $h = 1.8$, the phase portrait is shown below. There are two equilibria, $F_e = 10$, which is unstable, and $F_e = 90$, which is stable and the carrying capacity.



b. When $h = 5$, the only equilibrium is $F_e = 50$. The phase portrait is shown below. For $h > 5$, there are no equilibria, and the population goes extinct.