1. The parabola is symmetric about the $y$-axis and is show in the diagram below. The point $(x, y)$ in the diagram lies on the parabola and appears in the upper right corner of the rectangle. By the symmetry, we see that the area of the rectangle is $A = 2xy$, where $y$ satisfies $y = 12 - x^2$. It follows that we can write

$$A(x) = 2x(12 - x^2) = 24x - 2x^3.$$ 

Differentiating this expression gives

$$A'(x) = 24 - 6x^2.$$ 

The maximum occurs when the derivative is zero, so $24 - 6x^2 = 0$ or $x^2 = 4$. Thus, $x = 2$ and $y = 12 - 2^2 = 8$. From the diagram, it follows that the dimensions of the largest rectangle inscribed in the parabola has a width of 4 and a height of 8. ($-2 \leq x \leq 2$, $0 \leq y \leq 8$) This gives the maximum area as $A_{\text{max}} = 4 \cdot 8 = 32$.

2. Begin this problem by drawing a diagram as shown below. The area of the study plot is $A = xy$ with the fence enclosing the region having length $P = 2x + y$, where $x$ is perpendicular to the river and $y$ is parallel to the river.

Since there is 20 m of fence, $2x + y = 20$ or $y = 20 - 2x$. The area of the region satisfies

$$A(x) = x(20 - 2x) = 20x - 2x^2.$$ 

Differentiating $A(x)$ yields

$$A'(x) = 20 - 4x,$$
which is zero when \(20 - 4x = 0\) or \(x = 5\). It follows that \(y = 20 - 2(5) = 10\). Thus, the maximum study area has 5 m of fence perpendicular to the river and 10 m of fence parallel to the river with a maximum area of \(A_{\text{max}} = 50\ \text{m}^2\).

3. Below is a diagram of the rectangular box described in the problem.

The volume of the box is given by \(V = lwh = (2w)wh\). The surface area of the open box satisfies
\[
S = lw + 2lh + 2wh = 2w^2 + 4wh + 2wh = 2w^2 + 6wh = 600\text{in}^2.
\]
This equation is solved giving \(wh = \frac{600 - 2w^2}{6} = 100 - \frac{w^2}{3}\) or \(h = \frac{100}{w} - \frac{w}{3}\). This is substituted into the equation for the volume of the box resulting in
\[
V(w) = 2w \left(100 - \frac{w^2}{3}\right) = 200w - \frac{2}{3}w^3.
\]
Differentiating \(V(w)\) gives
\[
V'(w) = 200 - 2w^2.
\]
Setting the derivative equal to zero gives \(2w^2 = 200\) or \(w^2 = 100\), so \(w = 10\) in (the width of the box). Thus, the length is \(l = 2w = 20\) in. The height is \(h = \frac{100}{10} - \frac{10}{3} = \frac{20}{3}\) in. The maximum volume is \(V_{\text{max}} = 20 \cdot 10 \cdot \frac{20}{3} = 4000/3\ \text{in}^3\).

5. To the right is a diagram of the can for this problem

The volume of the can satisfies
\[
V = \pi r^2 h = 1000\ \text{cm}^3.
\]
The surface area of the can is given by
\[
S = 2\pi rh + 2\pi r^2\ \text{cm}^2,
\]
since it consists of the lateral side and two circular ends. The condition on the volume gives the height
\[
h = \frac{1000}{\pi r^2}.
\]
Substituting this expression into the equation for the surface area gives
\[
S(r) = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2 = 2000r^{-1} + 2\pi r^2.
\]
Differentiating this expression yields
\[
S'(r) = -2000r^{-2} + 4\pi r = \frac{4\pi r^3 - 2000}{r^2}.
\]
The derivative is zero at the minimum when the numerator of the expression above is zero, so

\[ 4\pi r^3 - 2000 = 0 \quad \text{or} \quad r^3 = \frac{1000}{2\pi}. \]

By taking the cube root of both sides we have the radius of the can is

\[ r = \frac{10}{\sqrt[3]{2\pi}} \approx 5.419 \text{ cm}. \]

The height satisfies

\[ h = \frac{1000}{\pi \left( \frac{10}{\sqrt[3]{2\pi}} \right)^2} = 10\sqrt[3]{\frac{4}{\pi}} \approx 10.839 \text{ cm}. \]

6. The beam with width \( w \) and depth \( d \) that is cut from a circular log with a radius of \( r \) is shown in the diagram below. It satisfies the equation

\[
\left( \frac{w}{2} \right)^2 + \left( \frac{d}{2} \right)^2 = r^2 \quad \text{or} \quad w^2 + d^2 = 4r^2.
\]

With the constant \( k \) for the proportionality constant, then the strength of the beam is \( S = kwd^2 \). From the expression above, \( d^2 = 4r^2 - w^2 \), so

\[ S(w) = kw(4r^2 - w^2) = k(4r^2w - w^3). \]

To find the strongest beam, we differentiate \( S(w) \), then set the derivative equal to zero. Thus,

\[ S'(w) = k(4r^2 - 3w^2) = 0. \]

So the width of the beam satisfies

\[ 3w^2 = 4r^2 \quad \text{or} \quad w = \frac{2r}{\sqrt{3}}, \]

and the depth of the beam is given by

\[ d^2 = 4r^2 - \frac{4}{3}r^2 = \frac{8}{3}r^2 \quad \text{or} \quad d = 2r \sqrt{2/3}. \]

8. a. The maximum value of \( P(c) \) occurs when the derivative of \( P \) with respect to \( c \) is zero. We use the quotient rule, so

\[ P'(c) = \frac{1000(1 + 100c^2) - 1000c(200c)}{(1 + 100c^2)^2} = \frac{1000(1 - 100c^2)}{(1 + 100c^2)^2}. \]
This is zero when the numerator is zero, so $1000(1 - 100c^2) = 0$ or $100c^2 = 1$. This gives $c^2 = \frac{1}{100}$ or $c = 0.1$. Since $P(0.1) = \frac{1000(0.1)}{1+100(0.1)^2} = 50$, we have that the optimal concentration is $c = 0.1$ M, and the maximal population density is $P_{\text{max}} = 50$ organisms/cm$^2$.

b. Note the graph passes through the origin and has a horizontal asymptote of $P = 0$ for large values of $c$ (as described by both the problem and shown by the function). Below is the graph of this function

![Graph of the function](image)

10. a. The model for age-structured populations is given by

$$r(x) = \frac{\ln(e^{-ax}bx^c)}{x} = \frac{\ln(e^{-ax}) + \ln(b) + c\ln(x)}{x} = \frac{-ax + \ln(b) + c\ln(x)}{x}.$$  

From the quotient rule, the derivative is given by

$$r'(x) = \frac{x(-a + c/x) - (-ax + \ln(b) + c\ln(x))}{x^2} = \frac{c - \ln(b) - c\ln(x)}{x^2}.$$  

The optimal rate of increase is assumed to be the maximum rate. Setting the derivative equal to zero, we take only the numerator from the derivative above. Thus,

$$c - \ln(b) - c\ln(x) = 0 \quad \text{or} \quad \ln(x) = 1 - \frac{\ln(b)}{c}.$$  

Exponentiating

$$x = e^\left(1 - \frac{\ln(b)}{c}\right) = e \cdot b^{-1/c}.$$  

Thus, the optimal age of reproduction is $x = e \cdot b^{-1/c}$. For parameters in Part b, this is $x = 0.582$.

b. For the graphs, see the short solutions.