1. Find the area of the largest rectangle with a base on the x-axis and the upper vertices on the parabola $y = 12 - x^2$. Give the dimensions of this rectangle.

2. A rectangular study plot is bounded on one side by a river, and the other three sides are to be blocked off by a fence. Find the dimensions of the plot that maximizes the area enclosed with 20 meters of fence.

3. An open box with its base having a length twice its width is to be constructed with 600 in^2 of material. Find its dimensions that maximize the volume.

4. Find the dimensions of an open rectangular box with a square base that holds 32 in^3 and is constructed with the least building material possible.

5. Find the dimensions of a right circular cylindrical can with both a top and a that holds one liter (1000 cm^3) and is constructed with the least amount of material possible.

6. The strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from a circular log with a radius of r.

7. A catalyst for a chemical reaction is a substance that controls the rate of the chemical reaction without changing the catalyst itself. An autocatalytic reaction is one whose product is a catalyst for its own formation.

$$A + X \xrightarrow{k} X.$$

The rate of this reaction $v = dx/dt$ is given by the formula

$$v = kx(a - x),$$

where a is the initial concentration of the substance A, x is the concentration of the product X, and k is the rate constant of the reaction. Find the concentration x that produces the maximum rate of reaction.

8. Nutrients in low concentrations inhibit growth of an organism, but high concentrations are often toxic. Let c be the concentration of a particular nutrient (in moles/liter) and P be the population density of an organism (in number/cm^2). Suppose that it is found that the effect of this nutrient causes the population to grow according to the equation:

$$P(c) = \frac{1000c}{1 + 100c^2}.$$

a. Find the concentration of the nutrient that yields the largest population density of this organism and what the population density of this organism is at this optimal concentration.

b. Sketch a graph of the population density of this organism as a function of the concentration of the nutrient.

9. One question for fishery management is how to control fishing to optimize profits for the fishermen. We will soon study the continuous logistic growth equation for populations. One differential equation describing the population dynamics for a population of fish F with harvesting is given by the equation,

$$\frac{dF}{dt} = rF\left(1 - \frac{F}{K}\right) - xF,$$

where r is the growth rate of this species of fish at low density, K is the carrying capacity of this population, and x is the harvesting effort of the fishermen. We will show that the non-zero equilibrium of this equation is given by

$$F_e = K\frac{(r-x)}{r}.$$

One formula for profitability is computed by the equation

$$P = xF_e,$$

so

$$P(x) = Kx\frac{(r-x)}{r}.$$

Find the maximum profit possible with this dynamics. What is the equilibrium population at this optimal profitability? Also, determine the maximum possible fish population for this model and at what harvesting level this occurs. (Clearly, this is a grossly oversimplified model, but can give some estimates for long range management.)

10. (From [1]) Semelparous organisms breed only once during their lifetime. Examples of this type of reproduction strategy can be found with Pacific salmon and bamboo. The per capita rate of increase, r, can be thought of as a measure of reproductive fitness. The greater r, the more offspring an individual produces. The intrinsic rate of increase is typically a function of age, x. Models for age-structured populations of semelparous organisms predict that the intrinsic rate of increase as a function of x is given by

$$r(x) = \frac{\ln[l(x)m(x)]}{x},$$

where $l(x)$ is the probability of surviving to age x and $m(x)$ is the number of female births at age x. Suppose that

$$l(x) = e^{-ax}$$

and

$$m(x) = bx^c,$$

where a, b, and c are positive constants.

a. Find the optimal age of reproduction.

b. Sketch graphs of $l(x)$, $m(x)$, and $r(x)$ for $a = 0.1$, $b = 4$, and $c = 0.9$.

[1] D. A. Roff, *The Evolution of Life Histories*, Chapman & Hall, 1992.

11. A female otter hears the cries of distress from her young in a den across and up the river from where she is foraging. (See the diagram below.) Assume that she is initially at Point A with the den residing at Point C. She wants to reach her young in the minimum amount of time. Assume she can run along the bank at $v_1 = 10$ ft/sec and swim through the river as $v_2 = 6$ ft/sec. The river is 200 ft wide and the den is 500 ft up the river. (We are ignoring the current in the river.) If the distance she runs along the bank (from A to B) is d_1 and the distance she swims (from B to C) is d_2, then the time for her to reach the den is given by the formula

$$T = \frac{d_1}{v_1} + \frac{d_2}{v_2}.$$

a. Use the diagram below to form an expression for the time as a function of x (the distance downstream from the den, where she crosses), $T(x)$.

b. Use your expression for the time $T(x)$ to find the minimum time for the otter to reach her pups. Give both the distance x and the time at the minimum.