Solve the following linear differential equations:

1. $\frac{dy}{dt} = 2y, \quad y(0) = 6.$

2. a. $\frac{dz}{dt} = 0.1z - 2, \quad z(0) = 5.$

3. $\frac{dx}{dt} = -\frac{x}{3}, \quad x(0) = 10.$

2. b. $\frac{dh}{dx} = 5 - 0.2h, \quad h(0) = 10.$

4. $\frac{dy}{dt} = 0.02y, \quad y(2) = 50.$

5. $\frac{dr}{dt} = 1 - \frac{r}{4}, \quad r(1) = 6.$

6. a. A population of yeast satisfies the differential equation for Malthusian growth. If this population satisfies

$$\frac{dY}{dt} = 0.14Y,$$

with an initial population of 100 and t is in hours, then determine its population as a function of time, and find at what time the population doubles.

b. A competing population is falling, due to the presence of this new population. Suppose it satisfies the initial value problem

$$\frac{dP}{dt} = -0.07P, \quad P(0) = 1000.$$

Find the solution to this problem, and determine when its population is half of its original population.

c. Find when the two populations are equal.

7. a. The population of Canada was 24,070,000 in 1980, while in 1990 it was 26,620,000. Assuming the population is growing according to the principle of Malthusian growth (with no food or space limitations), find the population as a function of time, and determine its doubling time.

b. For the same years, the populations of Kenya were 16,681,000 and 24,229,000, respectively. Find the population of Kenya as a function of time, assuming it too is growing with Malthusian growth. What is Kenya's doubling time for its population?

c. Use these models to project the populations in the two countries in the year 2000. In what year do the populations of Canada and Kenya become equal?

A. A radioactive substance satisfies the differential equation

$$R'(t) = -kR(t)$$

for some constant k. Suppose that initially there are 10 mg of the substance. After 25 days there are 8 mg remaining. Find k and determine the half-life of the substance (time when $R(t) = 5$).

8. When Strontium-90 (^{90}Sr) is ingested, it can displace calcium in the formation of bones. After a beta decay, it becomes an isotope of krypton (an inert gas), and diffuses out of the bone, leaving the bones porous.

a. Suppose that a particular bone contains 20 mg of ^{90}Sr, which has a half-life of 28 years. Write an equation describing the amount of ^{90}Sr remaining at any time, and determine the amount after 10 years.

b. Find how long until only 7 mg of ^{90}Sr remain.

B. Excessive salt consumption or kidney failure leads to a build up of water in the blood. (Physiologically, one complains of feeling bloated or having decreased urination.) The retention of water in the blood increases the total blood volume, which in turn, increases the *stroke volume*, V.

a. Assume that the stroke volume increases by about 20% over normal, so that

$$V = 0.1 \text{ liter}$$

Using a normal pulse rate of 70 beats/min, determine how this affects the cardiac output Q and the blood pressure readings, assuming that the resistance, $R_s = 17.6$ (mm Hg/liter/min) and the compliance $C_a = 0.002$ (liters/mm Hg) are at normal levels.

b. The body will adapt to this increased retention of water by increasing the resistance, R_s, to return the systemic blood flow to normal. Assume that the systemic resistance increases by about 20%, so

$$R_s = 21.1 \text{ mm Hg/liter/min}.$$

If the compliance retains its normal value and the stroke volume has the same value as in Part a, then compute what happens to the systolic and diastolic blood pressure readings.

C. Arteriosclerosis is a disease characterized by a build up of cholesterol and fatty deposits (plaque) within the walls of the arteries. This results in what has been called "hardening of the arteries." The result is that the arteries lose their elasticity or the compliance of the arteries drops.

a. Assume that the compliance decreases by about 20% over normal, so that

$$C_a = 0.0016 \text{ liters/mm Hg}.$$

Assume a normal pulse rate of 70 beats/min, cardiac output of $Q = 5.6$ liters/min, and resistance, $R_s = 17.6$ (mm Hg/liter/min). Determine how this affects the systolic and diastolic blood pressure readings.

b. When sufficient buildup occurs, the arterial wall can rupture. The result is that calcified plaque is exposed to the blood stream and platlets adhere to the walls of the arteries. This narrows the arteries and increases the resistance. Use the same values as in Part a., except for an increase in the systemic resistance increases by about 20%, so

$$R_s = 21.1 \text{ mmHg/liter/min}.$$

Compute the changes to the systolic and diastolic blood pressure readings.

9. You have just boiled a new batch of broth for your important culture of *E. coli*, so it is at 100°C. You have it sitting in a room that is at 22°C, and you find 5 minutes later that it's cooled to 93°C. You want to innoculate the culture when it reaches 40°C. You are interested in knowing if you can safely go off to exercise while the broth cools.

a. Let $T(t)$ be the temperature of the broth. Assume that the broth satisfies Newton's Law of Cooling, and set up the differential equation for the temperature of the broth, and solve it.

b. Find how long it will be until you need to innoculate the broth with your culture. Sketch a graph of the function of $T(t)$ for the first hour showing its starting and ending temperatures for the hour.

10. A thin plate is heated to 100°C for purposes of sterilization. Assume that the plate is placed into a room at 20°C and cools according Newton's Law of cooling, *i.e.*, the change in temperature of the plate is proportional to the difference between the temperature of the room and the temperature of the plate. After 10 min the plate is found to have a temperature of 80°C.

a. Write the differential equation describing the temperature of the plate, $T(t)$, and solve it for any time $t \geq 0$.

b. Find when the plate has cooled to 30°C so that it can be innoculated with a culture of cells.

11. a. *Paramecia* in a pond sample are growing according to the principle of Malthusian growth (with no food or space limitations). Initially, there are 1500 *Paramecia*. Four hours later, the population has 2000 individuals. Find the population of *Paramecia* as a function of time, and determine its cell doubling time.

b. A large population of 5000 is transferred to where a limited diet affects their growth dynamics. The *Paramecia* now satisfy the population dynamics of the differential equation:

$$\frac{dP}{dt} = -0.1P + 100.$$

Solve this differential equation and find what happens to the population as $t \to \infty$. (Hint: Recall your techniques for Newton's Law of cooling.)

12. A well mixed pond, $V = 200,000$ m^3, is initially clean $(c(0) = 0)$. A polluted stream with a concentration of dioxin, $Q = 5$ ppb enters the pond, flowing at a rate of $f = 4000$ m^3/day. Another stream carries the water away at the same rate.

a. Set up the initial value problem and solve it.

b. Find how long before the pond has a concentration of 4 ppb.

c. Find the limiting concentration.

D. a. Suppose that an initially clean lake $(c(0) = 0)$ with a constant volume of $V = 10^6$ m^3 has a stream flowing in at $f = 2500$ m^3/day. If the stream contains dioxin at a concentration of $Q = 12$ ppb and if you are given that the differential equation describing this concentration of pollutant is given by

$$c' = \frac{f}{V}(Q - c),$$

then solve this differential equation.

b. Determine how long until the lake has a concentration of 10 ppb of dioxin. Find the limiting concentration of dioxin.

c. If the lake achieves the limiting concentration, then regulations shut down all pollutants entering the lake from the stream $(Q = 0)$. Find how long it takes for the lake to decrease to only 4 ppb of dioxin.

13. a. Suppose that an initially clean lake $(c(0) = 0)$ with a constant volume of 3000000 m^3 has one stream flowing in at $f = 4000$ m^3/day. This stream is found to contain a pollutant at a concentration of $Q = 18$ ppb. There is another stream flowing in at $f = 2500$ m^3/day, and this stream is found to contain the same pollutant at a concentration of $Q = 4$ ppb. The lake is well-mixed and water leaves at the same rate as it flows in from the two feeding streams. Set up the differential equation for the concentration of the pollutant in the lake. (Write this differential equation in the form $\frac{dc}{dt} = A - Bc$.) Solve this differential equation.

b. Determine how long until the lake has a concentration of 4 ppb of the pollutant. Find the limiting concentration of the pollutant.

14. Denote by $L(t)$ the length of a fish at time t and assume that the fish grows according to the von Bertalanffy equation

$$\frac{dL}{dt} = k(34 - L(t)) \quad L(0) = 2.$$

a. Solve this differential equation.

b. Use your solution from Part a to determine k under the assumption that $L(4) = 10$. Sketch the graph of $L(t)$ for this value of k.

c. Find the length of the fish when $t = 10$. Also, find the asymptotic length of the fish, *i.e.*, find $\lim_{t\to\infty} L(t)$.