**Homework Solutions – Due Mon. 10/04/2018**

**Proof Problem**. (10pts) a. Consider the IVP:

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

In a closed, bounded region containing $(t_0, y_0)$, it can be shown that this has a unique solution, $\phi(t)$. By Taylor's Theorem, it has a solution:

$$\phi(t_{n+1}) = \phi(t_n) + h\phi'(t_n) + \frac{1}{2}\phi''(\bar{t}_n)h^2, \quad \text{with} \quad \bar{t}_n \in [t_n, t_n + h].$$

Euler's formula gives:

$$y_{n+1} = y_n + hf(t_n, y_n).$$

If we define $E_n = \phi(t_n) - y_n$ and note that $\phi'(t) = f(t, \phi)$, then

$$\begin{aligned} E_{n+1} &= \phi(t_{n+1}) - y_{n+1}, \\ &= \phi(t_n) + hf(t_n, \phi(t_n)) + \frac{1}{2}\phi''(\bar{t}_n)h^2 - y_n - hf(t_n, y_n), \\ &= E_n + h\left(f(t_n, \phi(t_n)) - f(t_n, y_n)\right) + \frac{1}{2}\phi''(\bar{t}_n)h^2. \end{aligned}$$

Since $f$ is assumed to satisfy the Lipschitz condition $|f(t, y) - f(t, \bar{y})| < L|y - \bar{y}|$ and $\beta = \max_{t_0 \le t \le t_{n+1}} |\phi(t)|/2$, then we have

$$\begin{aligned} |E_{n+1}| &\le |E_n| + h\,|f(t_n, \phi(t_n)) - f(t_n, y_n)| + \beta h^2, \\ &\le |E_n| + hL\,|\phi(t_n) - y_n| + \beta h^2, \\ &\le \alpha|E_n| + \beta h^2, \end{aligned}$$

where $\alpha = 1 + hL$.

b. If $E_0 = 0$, then $|E_1| \le \beta h^2$. Continuing

$$\begin{aligned} |E_2| &\le \alpha|E_1| + \beta h^2 \le \beta h^2 (1 + \alpha), \\ |E_3| &\le \alpha|E_2| + \beta h^2 \le \beta h^2 \left(1 + \alpha + \alpha^2\right), \\ &\vdots \qquad \vdots \\ |E_n| &\le \beta h^2 \sum_{i=0}^{n-1} \alpha^i. \end{aligned}$$

This is a finite geometric series, which was shown in Calculus to satisfy:

$$|E_n| \le \beta h^2 \sum_{i=0}^{n-1} \alpha^i = \beta h^2 \frac{\alpha^n - 1}{\alpha - 1}.$$

From the definition of $\alpha = 1 + hL$, it is obvious that

$$|E_n| \le \beta h \frac{(1 + hL)^n - 1}{L}.$$

This error is clearly increasing exponentially in $n$.

c. Consider the function $g(x) = \ln(1+x)$ for $x > 0$, then $g'(x) = \frac{1}{1+x} < 1$. It follows that $\ln(1+x) \le x$ for all $x \ge 0$. Thus,

$$\begin{aligned} \ln(1+hL) &\le hL, \\ \ln(1+hL)^n &\le nhL, \\ (1+hL)^n &\le e^{nhL}. \end{aligned}$$

Thus,

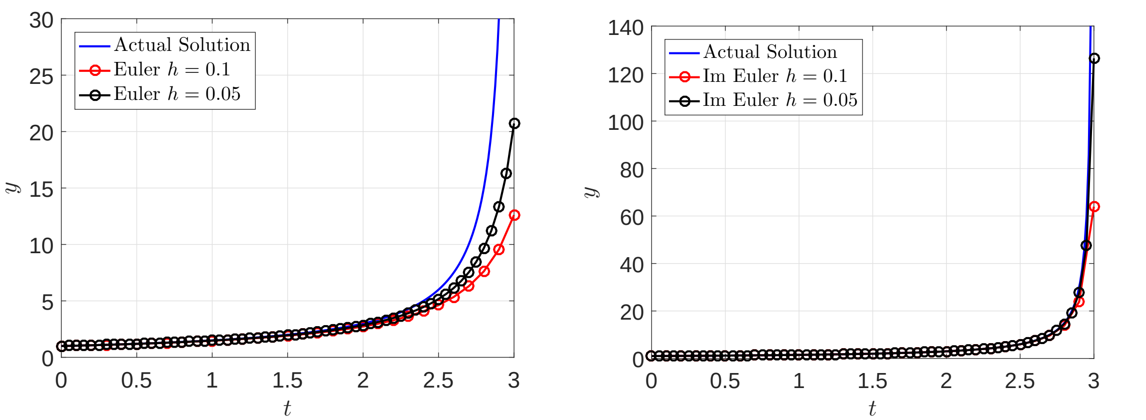
$$|E_n| \le \beta h \frac{e^{nhL}-1}{L}.$$

For $T > t_0$ and taking $h$ such that $n$ steps are required to reach $T$, then $nh = T - t_0$, so

$$|E_n| \le \beta h \frac{e^{(T-t_0)L}-1}{L} = Kh,$$

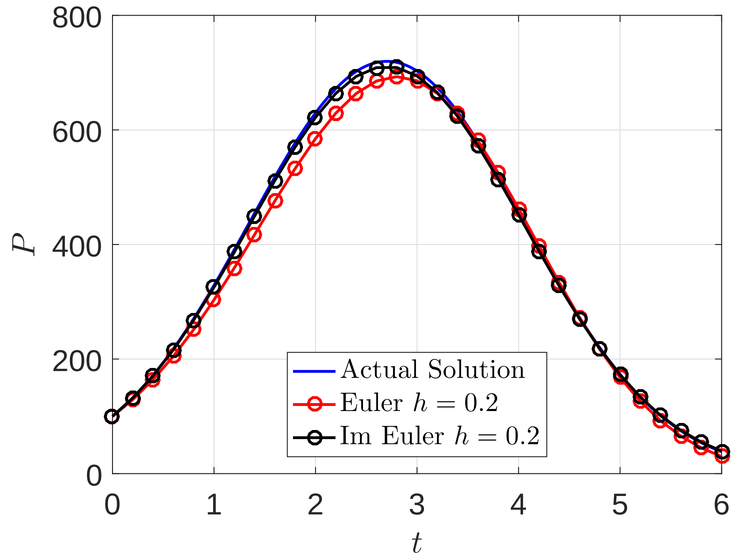
where $K$ depends on the length of the interval and the constants $L$ and $\beta$, which came from properties of $f$.

8. (9pts) c. The differential equation, $y' = y^2/3$, has a vertical asymptote for finite $t$ ($t = 3$) depending on the initial condition, $y(0) = 1$. (Different versions have different asymptotes.) The solutions track well for the early part of the interval, but lose accuracy as $t$ approaches the asymptote. The smaller stepsizes improve the computations. However, Improved Euler's method does much better at tracking the actual solution for a longer time.




Euler simulation　　Improved Euler simulation

f. The graph of this differential equation ($P' = (1.46 - 0.54\,t)P$) again shows that the Improved Euler's method is significantly better at tracking the actual solution. Thus, the maximum is much better tracked by the Improved Euler's method with a fair amount of error seen for Euler's method. Changing to Improved Euler's method is better than decreasing the stepsize.



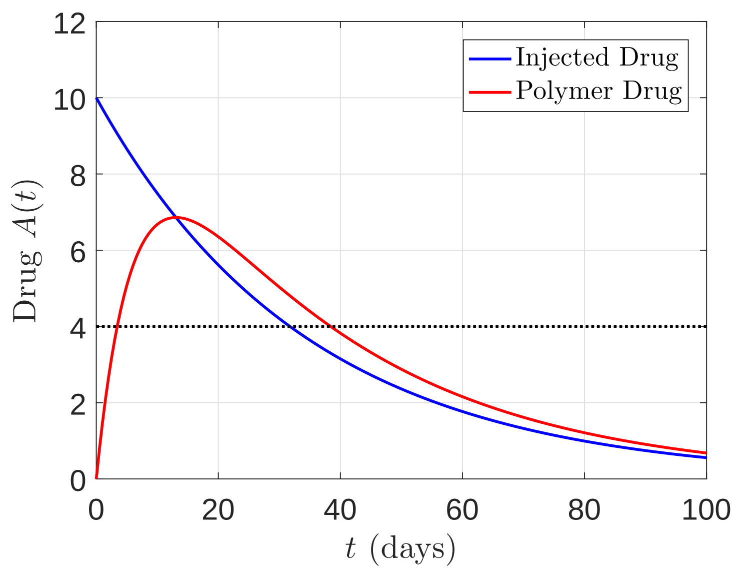

9. (7pts) c. The solution to the injected drug is:

$$A_i(t) = A_0 e^{-kt},$$

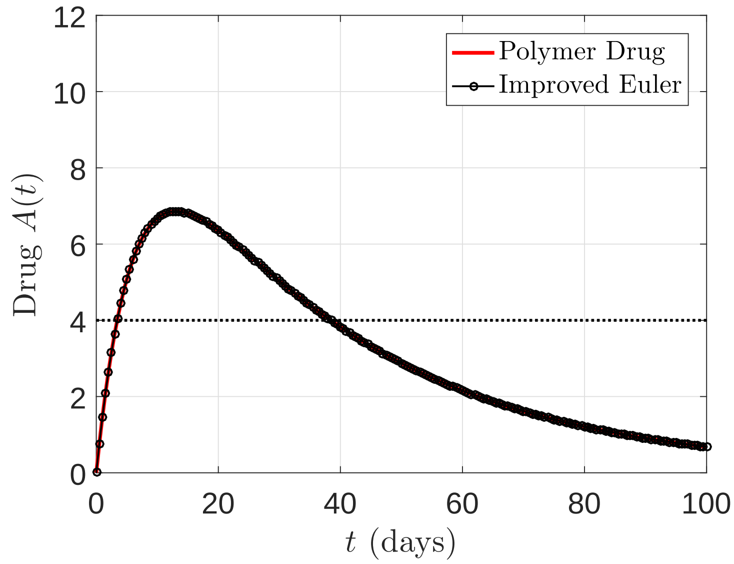
where $k = \ln(2)/t_h$ with $t_h$ being the half-life of the drug. The graphed example has $A_0 = 10$ and $t_h = 24$. The polymer released drug satisfies:

$$A_p(t) = \left(\frac{r}{q-k}\right)\left(e^{-kt} - e^{-qt}\right),$$

where $k$ is from before and $q$ and $r$ are specified. The graphed example has $q = 0.16$ and $r = 1.6$ From the graph it is clear that $A_p$ has the longer effective period and does not reach as high a concentration, so is superior as a drug delivery system.




e. The graph shows that the Improved Euler method matches very closely the actual solution.

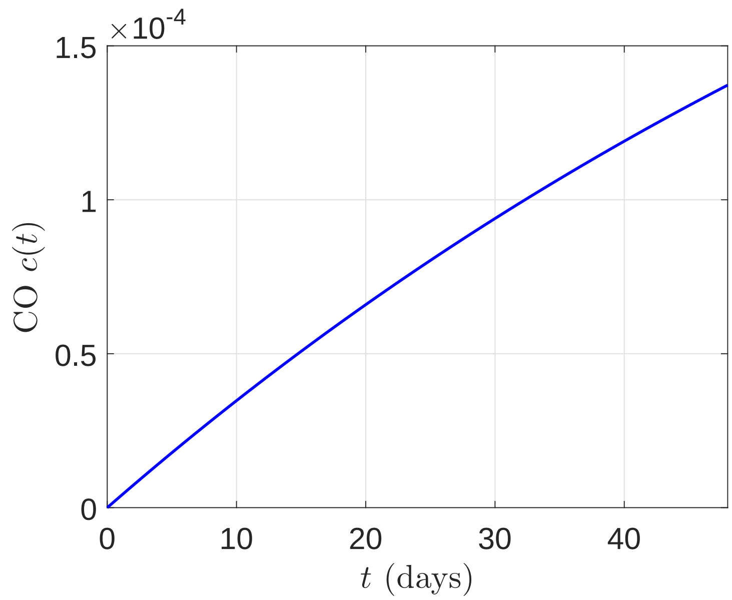



10. (7pts) b. The differential equation is given by:

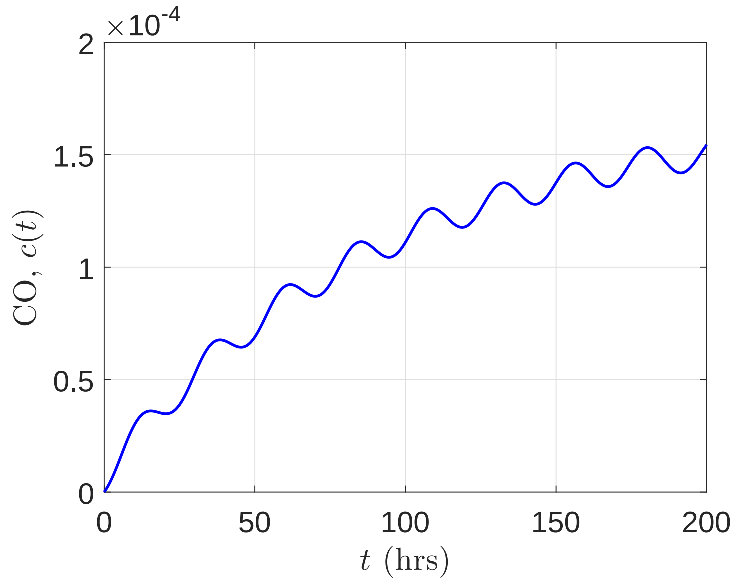
$$\frac{dc}{dt} = \frac{Q}{V} - \frac{f}{V}c,$$

which is the concentration entering minus the concentration leaving. Entering is the amount produced $Q$ divided by $V$ to make a concentration, while leaving is flow rate, $f$, times concentration, $c$, divided by the total volume, $V$. This is readily solved with our linear technique using the integrating factor of $e^{ft/V}$. The solution satifies:

$$c(t) = \frac{Q}{f}\left(1 - e^{-ft/V}\right).$$




e. With the Improved Euler's solution, we see the CO growing in a oscillatory manner with the unsafe level for this example being reached in about 80 hrs. This model is not producing as much CO, so it grows more slowly and has the distinct cycles every 24 hr.



×10^-4
2
1.5
1
0.5
0
CO, c(t)
0
50
100
150
200
t (hrs)