April 4, 2018 Math 541 Name Key

Be sure to show all your work.

1. (35pts) a. We want to solve the linear system $Ax = b$. Consider the matrix A and vector b given by:

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 1 & -1 & \beta^2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 4 \\ \beta+1 \end{pmatrix}.$$

Gaussian elimination with **partial pivoting** is written compactly in the form:

$$PA = LU,$$

where L is a lower triangular matrix representing the row operations of the Gaussian elimination, U is an upper triangular matrix, and P shows the pivoting operations. Find L, U, and P for the matrix A above. (**Hint:** Since you will be using the b in later calculations, you may want to work with b (augmented matrix) in your Gaussian operations to simplify later computations and significantly save time.)

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 4 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & \beta^2 & \beta+1 \end{array}\right] \xrightarrow[R_3 - \frac{1}{2}R_1]{R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 4 \\ 0 & 1 & -\frac{5}{2} & -1 \\ 0 & -1 & \beta^2 - \frac{3}{2} & \beta - 1 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 0 & 3 & 4 \\ 0 & 1 & -\frac{5}{2} & -1 \\ 0 & 0 & \beta^2 - 4 & \beta - 2 \end{array}\right]$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & -1 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & -5/2 \\ 0 & 0 & \beta^2 - 4 \end{pmatrix}$$

2, 5, 5

b. In solving $Ax = b$ with the vector b above, determine which values of β give no solution to this system and which values produce infinitely many solutions. When there are infinitely many solutions, give those solutions.

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$\beta = -2$, No Solution

$$\beta = 2, \quad \left[\begin{array}{ccc|c} 2 & 0 & 3 & 4 \\ 0 & 1 & -5/2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad x_p = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \quad x_h = \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$$

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$$x = c\begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

Continue using

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 1 & -1 & \beta^2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 4 \\ \beta+1 \end{pmatrix}.$$

c. You are given that

$$A^{-1} = \frac{1}{2(\beta^2-4)} \begin{pmatrix} -3 & \beta^2-1 & -3 \\ 2\beta^2-3 & -\beta^2-1 & 5 \\ 2 & -2 & 2 \end{pmatrix} = 12.469 \begin{pmatrix} -3 & 3.0401 & -3 \\ 5.0802 & -5.0401 & 5 \\ 2 & -2 & 2 \end{pmatrix}$$

Let $\beta = 2.01$ and find the solution to $Ax = b$. Find $||x||_1$, $||x||_2$, and $||x||_\infty$ (the 1-norm, 2-norm (Euclidean), and ∞-norm (max)).

From Part a

$\beta^2 = 4.0401$

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$$\left[\begin{array}{ccc|c} 2 & 0 & 3 & 4 \\ 0 & 1 & -5/2 & -1 \\ 0 & 0 & 0.0401 & 0.01 \end{array}\right]$$

$x_3 = 0.2494$

$x_2 = \frac{5}{2}x_3 - 1 = -0.37656$

$x_1 = 2 - \frac{3}{2}x_3 = 1.6259$

2 $\|x\|_1 = 1.6259 + 0.3765 + 0.2494 = 2.2518$

2 $\|x\|_2 = (1.6259^2 + 0.3765^2 + 0.2494^2)^{1/2} = \sqrt{2.8475034} = 1.68745$

2 $\|x\|_\infty = 1.6259$

d. Let $\beta = 2.01$. Find $||A||_1$. Also, determine the condition number for A, $\kappa_1(A)$, using the 1-norm with this value of β.

3 Last column $\|A\|_1 = |-1| + |3| + |\beta^2| = 8.0401$

4 $\|A^{-1}\|_1 = 12.469(|-3| + |5.0802| + |2|) = 12.469(10.0802) = 125.69$

$K_1(A) = \|A\|_1 \cdot \|A^{-1}\| = 1010.56$

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2. (15pts) a. We saw matrix methods applied to chemical reactions with diffusion. Consider a discretized version (very crude) that satisfies $Au_e = b$ with

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 80 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using the **direct method** of Gaussian elimination, solve this equation to find the equilibrium chemical distribution, u_e.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 80 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 80 \\ 0 & -3 & 1 & 0 & -80 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_3 + \frac{R_2}{3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 80 \\ 0 & -3 & 1 & 0 & -80 \\ 0 & 0 & -\frac{8}{3} & 1 & -\frac{80}{3} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

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$u_{4e} = 0$, $-\frac{8}{3}u_{3e} = -\frac{80}{3} \Rightarrow u_{3e} = 10$, $-3u_{2e} + 10 = -80 \Rightarrow u_{2e} = 30$, $u_{1e} = 80$

$$u_e = \begin{pmatrix} 80 \\ 30 \\ 10 \\ 0 \end{pmatrix}$$

b. We saw the **Jacobi's iterative method**, where A was decomposed into a diagonal matrix $D = a_{ii}$, a strictly lower triangular matrix $L = a_{ij}$ with $i > j$, and a strictly upper triangular matrix $U = a_{ij}$ with $i < j$, so that $A = D - L - U$. By defining $T = D^{-1}(L+U)$ and $c = D^{-1}b$, then a convergent iterative scheme was produced by

$$u^{(k+1)} = Tu^{(k)} + c, \qquad k = 0, 1, ...,$$

where $u^{(0)}$ is an initial vector. Find T and c. Let $u^{(0)} = [0, 0, 0, 0]^T$ be the initial vector (clean solution) and find the first 3 iterates, $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$.

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$$D^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad L + U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad T = D^{-1}(L+U) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D^{-1}b = c = \begin{bmatrix} 80 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$u^1 = Tu^0 + c = \begin{pmatrix} 80 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u^2 = Tu^1 + c = \begin{pmatrix} 80 \\ 80/3 \\ 0 \\ 0 \end{pmatrix} \quad u^3 = Tu^2 + c = \begin{pmatrix} 80 \\ 80/3 \\ 80/9 \\ 0 \end{pmatrix}$$

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3. (40pts) a. Evaluate the following integral and find the exact value:

$$\int_0^4 5e^{-x/2}dx. = -10e^{-x/2}\Big|_0^4$$

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$$= 10 - 10e^{-2} = 8.646647$$

b. Use the Trapezoid rule with $h = 1$ to approximate the integral. Give the absolute error between this approximation and the exact value.

$$\int_0^4 5e^{-x/2}dx \approx \frac{1}{2}\left[5 + 10e^{-1/2} + 10e^{-1} + 10e^{-3/2} + 5e^{-2}\right]$$

$$= \frac{1}{2}\left[5 + 6.06531 + 3.67879 + 2.23130 + 0.67668\right]$$

$$= 8.82604$$

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$$\text{Error} = |8.82604 - 8.646647| = 0.17939$$

c. Use the Simpson's rule with $h = 1$ to approximate the integral. Give the absolute error between this approximation and the exact value.

$$\int_0^4 5e^{-x/2}dx \approx \frac{1}{3}\left[5 + 4\cdot 5e^{-1/2} + 10e^{-1} + 4\cdot 5e^{-3/2} + 5e^{-2}\right]$$

$$= \frac{1}{3}\left[5 + 12.13061 + 3.67879 + 4.46260 + 0.67668\right.$$

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$$= 8.64956$$

$$\text{Error} = |8.64956 - 8.646647| = 0.002915$$

Continue with the definite integral:

$$\int_0^4 5e^{-x/2}dx.$$

3. d. Suppose you reduced the stepsize to $h = 0.25$, which is about 4 times the function evaluations (too many steps to do by hand, so do **NOT** calculate this). Use your knowledge of the convergence of these methods and the absolute errors calculated above to estimate the absolute error for each method using this stepsize. (Recall that for a given problem $Abs\ Err \approx Mh^p$ for some M and p.)

Trapezoid is $\mathcal{O}(h^2)$ $\quad 0.17939 \cdot 1^2 \approx M \quad \therefore h = 1/4 \Rightarrow$ Abs Err $\approx \frac{0.17939}{16}$

≈ 0.01121

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Simpson is $\mathcal{O}(h^4)$ $\quad 0.002915\,(1)^2 \approx M \quad \therefore h = 1/4 \Rightarrow$ Abs Err $\approx \frac{0.002915}{4^4}$

$\approx 1.139 \times 10^{-5}$

e. In this part you use Gaussian quadrature with 2 points to approximate the integral. The Gauss points GP and Gauss weights WT are given below:

$$GP = [-0.577350, 0.577350] \times 2 + 2$$
$$WT = [1, 1].$$

Interval translate by 2 expand factor 2

$GP = [0.8453, 3.1547]$ $\quad WT = [2, 2]$

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$\int_0^4 5e^{-x/2}dx \approx 2 \cdot 5e^{-0.8453/2} + 2 \cdot 5e^{-3.1547/2}$

$= 6.55308 + 2.06522 = 8.618296$

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