1. Consider the function $f(x) = \frac{x^3 - \ln(x)}{1-x^2} + \frac{2}{x^2} = \frac{x^3-\ln(x)}{1-x^2} + 2x^{-2}$. Applying the quotient rule to the first part and the power rule to the second, we have:

$$\begin{aligned} f'(x) &= \frac{(1-x^2)(3x^2 - \frac{1}{x}) - (x^3 - \ln(x))(-2x)}{(1-x^2)^2} - 2 \cdot 2x^{-3} \\ &= \frac{(1-x^2)(3x^2-\frac{1}{x}) + 2x(x^3 - \ln(x))}{(1-x^2)^2} - 4x^{-3}. \end{aligned}$$

2. Consider the function $f(x) = \frac{x^2 - e^{-x}}{3x+1} + xe^{-x}$. Applying the product and quotient rule to this function, we have:

$$\begin{aligned} f'(x) &= \frac{(3x+1)(2x+e^{-x}) - (x^2-e^{-x})(3)}{(3x+1)^2} + (-xe^{-x} + 1 \cdot e^{-x}) \\ &= \frac{(3x+1)(2x+e^{-x}) - 3(x^2-e^{-x})}{(3x+1)^2} + (1-x)e^{-x}. \end{aligned}$$

3. Consider the function $f(x) = \frac{\sqrt{x}}{2+x} - \frac{1}{e^{3x}} = \frac{x^{\frac{1}{2}}}{2+x} - e^{-3x}$. From our rules of differentiation, we have

$$\begin{aligned} f'(x) &= \frac{(2+x)\left(\frac{1}{2}x^{-\frac{1}{2}}\right) - (1)(x^{\frac{1}{2}})}{(2+x)^2} - (-3)e^{-3x} \\ &= \frac{2+x-2x}{2(2+x)^2\sqrt{x}} + 3e^{-3x} \quad = \quad \frac{2-x}{2(2+x)^2\sqrt{x}} + 3e^{-3x} \end{aligned}$$

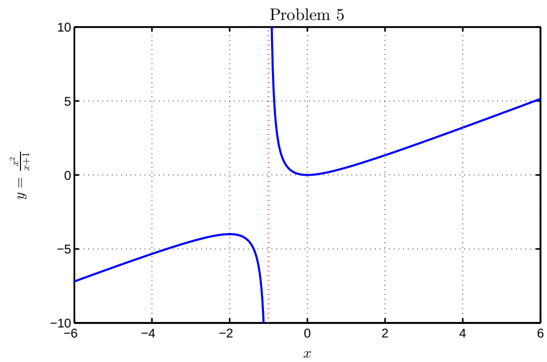
4. Consider the function $f(x) = \frac{8e^{-2x}}{12+\cos(2x)}$. The quotient rule gives the derivative

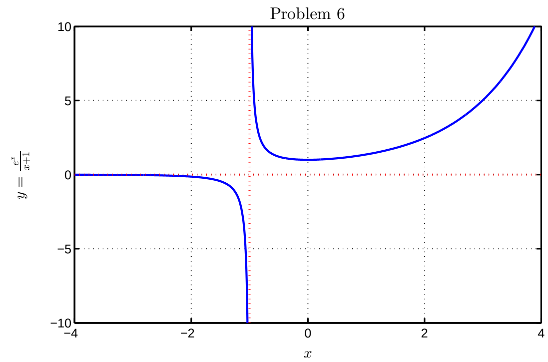
$$\begin{aligned} f'(x) &= \frac{(12+\cos(2x))(8(-2)e^{-2x}) - (8e^{-2x})(-2\sin(2x))}{(12+\cos(2x))^2} \\ &= \frac{16(\sin(2x) - \cos(2x) - 12)e^{-2x}}{(12+\cos(2x))^2}. \end{aligned}$$

5. Consider the function $y(x) = \frac{x^2}{x+1}$. The quotient rule finds the derivative

$$y'(x) = \frac{(x+1)(2x) - (1)x^2}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}$$

The x-intercept occurs where $y = 0$, which is $x = 0$, so the x and y-intercept is the origin, $(0,0)$. There is no horizontal asymptote, because the exponent in the numerator is higher than that in the denominator. The vertical asymptote occurs when the denominator is zero, or $x+1=0$ so $x=-1$. At the critical points, $y' = 0 = \frac{x(x+2)}{(x+1)^2}$, so $x(x+2) = 0$. Thus, $x_{1c} = -2$ and $x_{2c} = 0$. The y values are $y_{1c} = \frac{(-2)^2}{-2+1} = -4$ and $y_{2c} = \frac{0^2}{0+1} = 0$. It follows that $(-2,-4)$ is a maximum and $(0,0)$ is a minimum. The graph appears below to the left.





6. Consider the function $y(x) = \frac{e^x}{x+1}$. The quotient rule finds the derivative

$$y' = \frac{(x+1)e^x - (1)e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2}.$$

The x-intercept occurs when $y = 0$. Since the exponential function is not zero, there are no x-intercepts. The y-intercept occurs when $x = 0$ or $y(0) = \frac{e^0}{0+1} = 1$, so the y-intercept is $(0, 1)$. Since the denominator $x + 1 = 0$ when $x = -1$, this is a vertical asymptote. From the limit below,

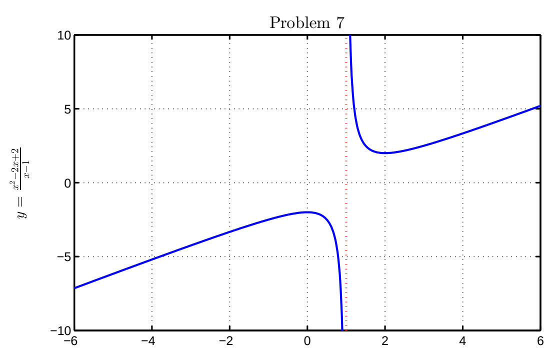
$$\lim_{x \to -\infty} \frac{e^x}{x+1} = 0,$$

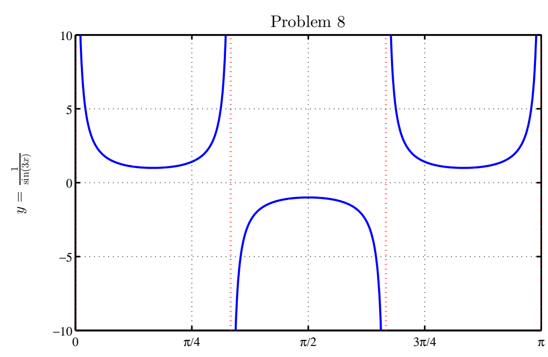
so there is a horizontal asymptote to the left at $y = 0$. The critical point satisfies $y' = 0$, so $0 = xe^x$ or $x_c = 0$. Since $y(x_c) = 1$, we have $(0, 1)$ is a minimum. The graph appears above on the right.

7. Consider the function $y(x) = \frac{x^2-2x+2}{x-1}$. The quotient rule finds the derivative

$$y' = \frac{(x-1)(2x-2) - (1)(x^2 - 2x + 2)}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}.$$

The x-intercept occurs when $y = 0$, but $x^2 - 2x + 2 = 0$ has no real solution, so there is no x-intercept. The y-intercept occurs when $x = 0$, so $y(0) = -2$. There is no horizontal asymptote, as the highest exponent in the numerator is larger than that in the denominator. There is a vertical asymptote where the denominator is zero, or $x = 1$. The critical points satisfy $y' = 0$, so $x(x-2) = 0$. It follows that $x_{1c} = 0$ and $y(x_{1c}) = -2$, which gives a maximum at $(0, -2)$. Similarly, $x_{2c} = 2$ and $y(x_{2c}) = 2$, which gives a minimum at $(2, 2)$. The graph is shown below on the left.





8. Consider the function $y = \frac{1}{\sin(3x)}$. Applying the quotient rule,

$$y'(x) = \frac{-3\cos(3x)}{\sin^2(3x)}.$$

The period is given by $T = \frac{2\pi}{\omega} = \frac{2\pi}{3}$. Critical points occur when $y' = 0$ or $\cos(3x) = 0$, which occurs when $3x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. Then $y(\frac{\pi}{6}) = \frac{1}{\sin(\frac{3\pi}{6})} = 1$ and $y(\frac{\pi}{2}) = \frac{1}{\sin(\frac{3\pi}{2})} = -1$. $(\frac{\pi}{6}, 1)$ is a minimum, and $(\frac{\pi}{2}, -1)$ is a maximum. The vertical asymptotes occur where $\sin(3x) = 0$ or $x = 0$, $\frac{\pi}{3}$, $\frac{2\pi}{3}$, and π. The curve of the function appears above on the right.

9. a. Consider the function $y(p) = \frac{p^4}{0.0625+p^4}$. The derivative satisfies

$$y'(p) = \frac{(0.0625 + p^4)(4p^3) - (p^4)(4p^3)}{(0.0625 + p^4)^2} = \frac{0.25p^3}{(0.0625 + p^4)^2}.$$

The second derivative can be found with a second application of the rule:

$$\begin{aligned} y'(p) &= \frac{0.25p^3}{(0.0625^2 + .125p^4 + p^8)} \\ y''(p) &= \frac{(0.0625 + p^4)^2(0.25 \cdot 3p^2) - (0.25p^3)(0.125 \cdot 4p^3 + 8p^7)}{(0.0625 + p^4)^4} \\ &= \frac{(0.0625 + p^4)^2(0.75p^2) - 0.25p^3(8p^3)(0.0625 + p^4)}{(0.0625 + p^4)^4} \\ &= \frac{0.25p^2(0.1875 - 5p^4)}{(0.0625 + p^4)^3}. \end{aligned}$$

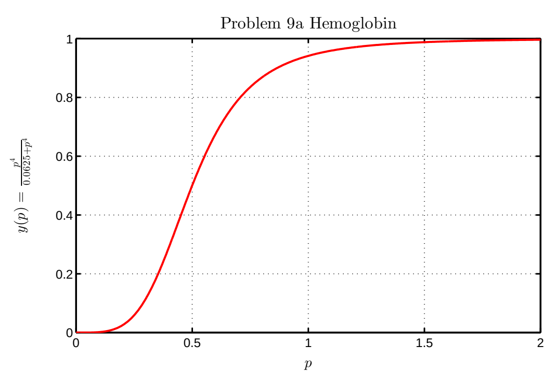
When $y''(p) = 0$, either $p = 0$ or $5p^4 = 0.1875$, giving $p_i \approx 0.44056$ for $p > 0$. At this point, $y(p_i) = \frac{0.0375}{0.0625+0.0375} = 0.375$, and $y'(p_i) = \frac{0.25(0.44056)^3}{(0.0625+0.0325)^2} = 2.13041$.

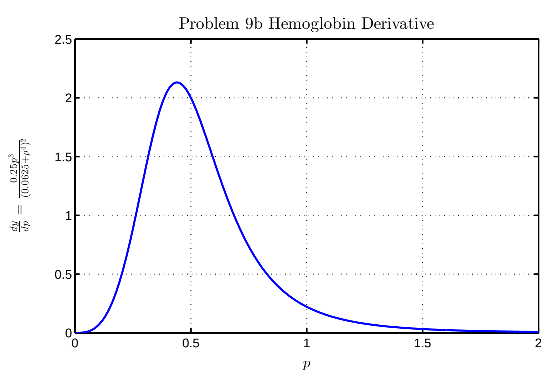
b. There is a y-intercept when $p = 0$, and $y(0) = 0$, so there is only one intercept at (0,0). There is a horizontal asymptote at $y = \frac{p^4}{p^4} = 1$. There is no vertical asymptote. The graph of $y(p)$ is shown below on the left. The graph of $y'(p)$ is shown below on the right.

c. The function reaches a 90% saturation when $y(p) = 0.9 = \frac{p^4}{0.0625+p^4}$. Thus,

$$0.9(0.0625 + p^4) = p^4 \qquad \text{or} \qquad p^4 = 0.9 \cdot 0.625 = 0.5625 \qquad \text{or} \qquad p = \sqrt[4]{0.5625} \approx 0.866025.$$

This curve is similar in shape to the O_2 dissociation curve, but the point of inflection occurs at $p = 21.2$ torr, which is about 50 times higher than the point of inflection for the CO dissociation curve, which implies that hemoglobin binds CO much more strongly than O_2.





10. a. The specific function is $R(L) = \frac{10L^2}{1+L^2}$. Differentiating this rate function using the quotient rule gives

$$\begin{aligned} R'(L) &= \frac{(1+L^2)10(2L) - (10L^2)(2L)}{(1+L^2)^2} \\ &= \frac{20L}{(1+L^2)^2} = \frac{20L}{(1+2L^2+L^4)}. \end{aligned}$$

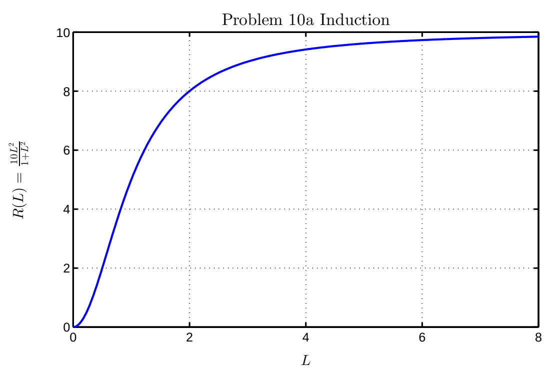
The second derivative satisfies:

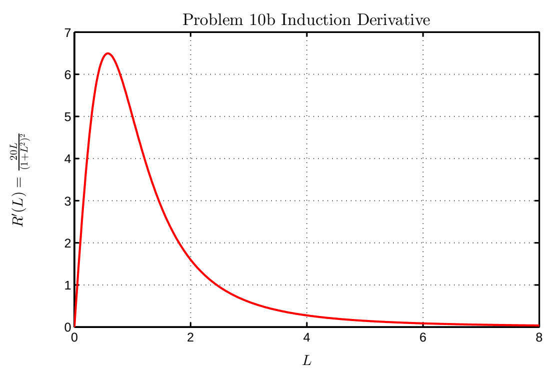
$$\begin{aligned} R''(L) &= \frac{(1+2L^2+L^4)(20) - (20L)(2\cdot 2L + 4L^3)}{(1+L^2)^4} \\ &= \frac{20(1+L^2)(1-3L^2)}{(1+L^2)^4} = \frac{20(1-3L^2)}{(1+L^2)^3} \end{aligned}$$

At the point of inflection, the second derivative is 0, when the numerator of the above expression is zero. Thus, $1 - 3L^2 = 0$ or $L = \frac{1}{\sqrt{3}} \approx 0.5774$. (Note we take only the positive square root because of the domain.) $R(\frac{1}{\sqrt{3}}) = \frac{10(\frac{1}{3})}{1+(\frac{1}{3})} = \frac{10}{4} = 2.5$. Thus, there is a point of inflection at (0.5774, 2.5). At this point, $R'(0.5774) = \frac{20L}{(1+L^2)^2} = \frac{20\cdot 0.5774}{(1+0.5774^2)^2} = 6.495$.

b. At the R-intercept, $L = 0$, so there is an intercept at (0,0). Since the highest power of L in the numerator and denominator is two, there is a horizontal asymptote at $y = \frac{10}{1} = 10$, using the leading coefficients of the highest powers. There is no vertical asymptote. The graph of $R(L)$ is shown below to the left.

c. The only intercept for the derivative is (0,0). The power of the denominator is greater than the power of L in the numerator, so the horizontal asymptote is $R'(L) = 0$. A sketch of $R'(L)$ is shown below to the right. Since the second derivative is zero at $L = 0.5774$, there is a maximum for $R'(L)$ at (0.5774, 6.495). Clearly, the L-value of the maximum matches the L-value for the point of inflection.





11. a. We consider the Beverton-Holt (Hassell's) model, which is given by $H(P) = \frac{5P}{1+0.004P}$. Differentiating using the quotient rule,

$$H'(P) = 5\frac{(1+0.004P) - P(0.004)}{(1+0.004P)^2} = \frac{5}{(1+0.004P)^2} = \frac{5}{1+0.008P+0.000016P^2}.$$

The second derivative is given by

$$H''(P) = \frac{0 - 5(0.008 + 2 \cdot 0.000016P)}{(1 + 0.004P)^4} = \frac{-0.04}{(1 + 0.004P)^3}.$$

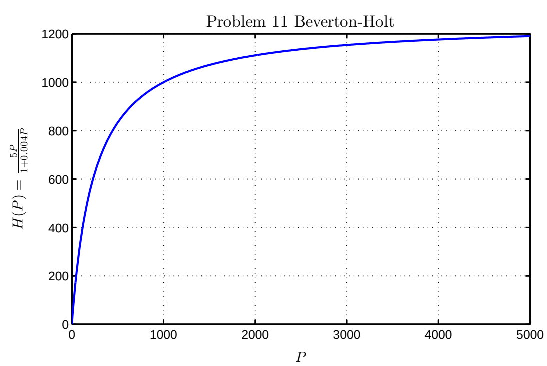
Note that $H''(P)$ is negative for $P \geq 0$.

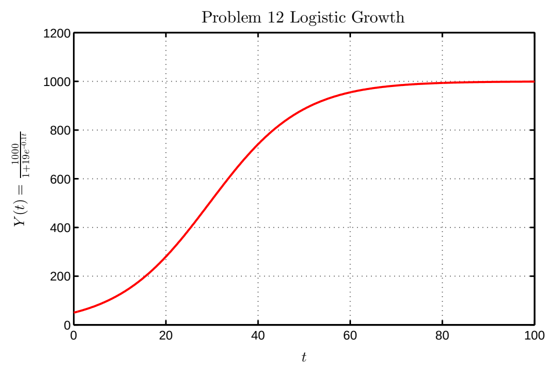
b. The P and H-intercept occurs when $P = 0$ and $H(0) = 0$. Since the leading powers of the numerator and denominator are the same (one), there is a horizontal asymptote at $H = \frac{5}{0.004} = 1250$. There is no vertical asymptote (for $P \geq 0$). A graph of $H(P)$ is shown below on the left.

c. Equilibria are found by solving

$$P_e = \frac{5P_e}{1 + 0.004P_e}.$$

This gives $P_e(1 + 0.004P_e) = 5P_e$, so there is the extinction equilibrium, $P_e = 0$, or $1 + 0.004P_e = 5$. This last equation gives the carrying capacity equilibrium, $P_e = 1000$.





12. a. Consider the logistic growth model, $Y(t) = \frac{1000}{1+19e^{-0.1t}}$. The derivative comes from the quotient rule,

$$Y'(t) = 1000\frac{(0 + 1.9e^{-0.1t})}{(1 + 19e^{-0.1t})^2} = \frac{1900e^{-0.1t}}{(1 + 19e^{-0.1t})^2} = \frac{1900e^{-0.1t}}{(1 + 38e^{-0.1t} + 361e^{-0.2t})}.$$

The second derivative is again found by the quotient rule

$$\begin{aligned} Y''(t) &= \frac{-190(1 + 38e^{-0.1t} + 361e^{-0.2t})e^{-0.1t} - 1900e^{-0.1t}(-3.8e^{-0.1t} - 72.2e^{-0.2t})}{(1 + 19e^{-0.1t})^4} \\ &= \frac{-190e^{-0.1t}(1 + 19e^{-0.1t})(1 - 19e^{-0.1t})}{(1 + 19e^{-0.1t})^4} \\ &= \frac{190e^{-0.1t}(19e^{-0.1t} - 1)}{(1 + 19e^{-0.1t})^3}. \end{aligned}$$

The second derivative is zero when $19e^{-0.1t} - 1 = 0$ or $e^{0.1t} = 19.$ or $t = 10\ln(19) \approx 29.44$. Then $Y(29.44) = \frac{1000}{1+19e^{-2.944}} \approx 500$. Thus, there is a point of inflection at $(29.44, 500)$.

b. There is a Y-intercept at $t = 0$, when $Y(0) = \frac{1000}{1+19} = 50$. The only intercept is $(0, 50)$. Since $\lim_{t\to\infty} e^{-0.1t} \to 0$,

$$\lim_{t\to+\infty} Y(t) \to 1000,$$

which gives a horizontal asymptote of $Y = 1000$. A sketch of $Y(t)$ is above to the right.

c. Since the population starts at 50, it doubles when it reaches 100. Solving $Y(t) = \frac{1000}{1+19e^{-0.1t}} = 100$ gives $1 + 19e^{-0.1t} = 10$, so $e^{0.1t} = \frac{19}{9}$. Thus, this population doubles when $t = 10\ln(\frac{19}{9}) \approx 7.47$ hr.

d. The Malthusian growth model doubles when it reaches 100. Solving $100 = 50e^{0.1t}$ gives $e^{0.1t} = 2$ or $t = 10\ln(2)$. Thus, the doubling time for the Malthusian growth model is $t \approx 6.93$ hr, which is less than for the logistic growth model.