1. The parabola is symmetric about the y-axis and is shown in the diagram below on the left. The point (x, y) in the diagram lies on the parabola and appears in the upper right corner of the rectangle. By the symmetry, we see that the area of the rectangle (**objective function**) is $A = 2xy$, where y satisfies $y = 12 - x^2$ (**constraint condition**). It follows that we can write

$$A(x) = 2x(12 - x^2) = 24x - 2x^3,$$

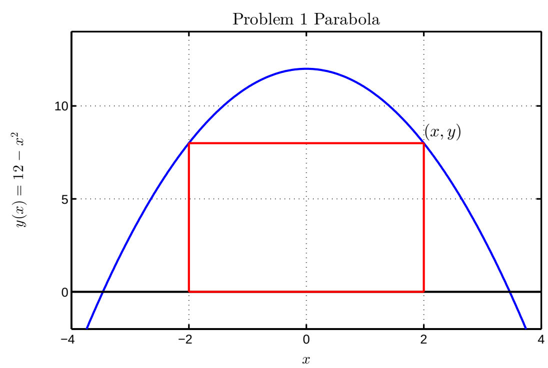
the **optimization problem**. Differentiating this expression gives

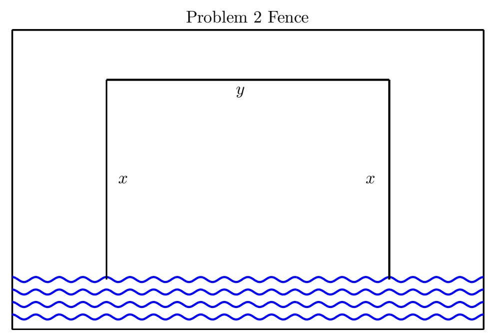
$$A'(x) = 24 - 6x^2 = -6(x^2 - 4).$$

The maximum occurs when the derivative is zero, so

$$x^2 - 4 = 0 \quad \text{or} \quad x = \pm 2.$$

Thus, the maximum occurs for $x_{max} = 2$, which gives $y_{max} = 12 - 2^2 = 8$. From the diagram, it follows that the dimensions of the largest rectangle inscribed in the parabola has a width of 4 and a height of 8 ($-2 \le x \le 2$ and $0 \le y \le 8$). This gives the maximum area as $A_{max} = 4 \cdot 8 = 32$.





2. Begin this problem by drawing a diagram as shown above on the right. The area of the study plot is $A = xy$ with the fence enclosing the region having length $P = 2x + y$, where x is perpendicular to the river and y is parallel to the river. Since there is 20 m of fence, the **constraint condition** satisfies:

$$2x + y = 20 \qquad \text{or} \qquad y = 20 - 2x.$$

The **optimization** or **objective function** is the area of the region, which satisfies

$$A(x) = x(20 - 2x) = 20x - 2x^2.$$

Differentiating $A(x)$ yields

$$A'(x) = 20 - 4x,$$

which is zero when $20 - 4x = 0$ or $x = 5$. It follows that $y(5) = 20 - 2(5) = 10$. Thus, the maximum study area has 5 m of fence perpendicular to the river and 10 m of fence parallel to the river with a maximum area of $A_{max} = 50$ m^2.

3. Below (left) is a diagram of the rectangular box described in the problem. The volume of the box (**objective function**) is given by $V = lwh = (2w)wh = 2w^2h$. The **constraint condition** in this problem is the surface area of the open box. The surface area of the open box satisfies

$$S = lw + 2lh + 2wh = 2w^2 + 6wh = 600 \text{ in}^2.$$

Solving for the variable h gives

$$6wh = 600 - 2w^2 \quad \text{or} \quad h = \frac{100}{w} - \frac{w}{3}.$$

This is substituted into the equation for the volume of the box resulting in

$$V(w) = 2w\left(100 - \frac{w^2}{3}\right) = 200w - \frac{2}{3}w^3,$$

which is the **optimizing function**. Differentiating $V(w)$ gives

$$V'(w) = 200 - 2w^2.$$

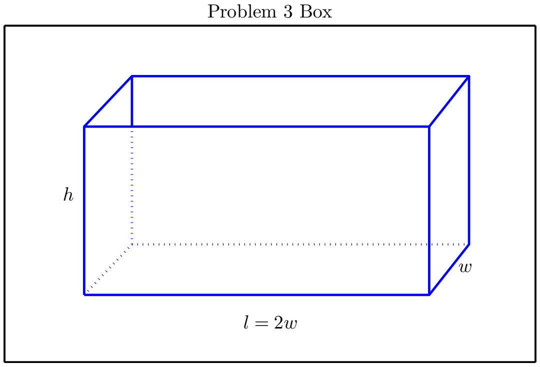
To find the optimal volume, we set the derivative equal to zero, so

$$2w^2 = 200 \quad \text{or} \quad w = \pm 10.$$

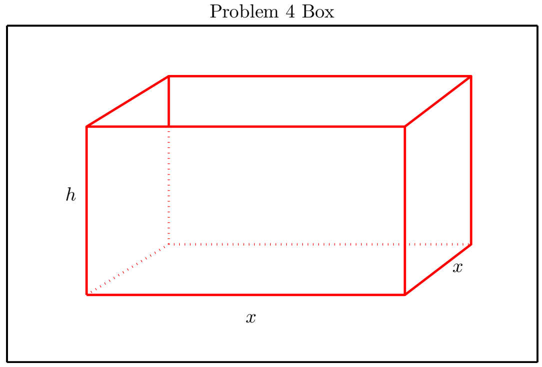
It follows that the optimal width of the box is $w = 10$ in. Thus, the length is $l = 2w = 20$ in. The height is $h = \frac{100}{10} - \frac{10}{3} = \frac{20}{3}$ in. The maximum volume is

$$V_{max} = 20 \cdot 10 \cdot \frac{20}{3} = \frac{4000}{3} \text{ in}^3.$$

Problem 3 Box



Problem 4 Box



4. Above right is a diagram of an open box with a square base, which has a volume satisfying $V = x^2h$, where h is the height of the box, and x is the side of the base. The **constraint condition** for this problem is $V = x^2h = 32$. Minimizing the construction materials means the surface area, S, is the least possible. The **objective function** is $S = x^2 + 4xh$, where we replace $h = \frac{32}{x^2}$. It follows that the surface area can be written

$$S(x) = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + 128x^{-1}.$$

To find the **optimal solution**, we differential $S(x)$ and obtain

$$S' = 2x - 128x^{-2} = \frac{2(x^3 - 64)}{x^2} = 0.$$

It follows that $x^3 - 64 = 0$ or the base length, $x = 4$ in produces the minimal surface area for the box. Substituting gives the height, $h = \frac{32}{4^2} = 2$ in, and the minimal surface area, $S_{min} = 4^2 + \frac{128}{4} = 48$ in^2.

5. Below on the left is a diagram of the can for this problem. The volume of the can satisfies

$$V = \pi r^2 h = 1000 \text{ cm}^3,$$

which is the **constraint condition**. The surface area of the can is given by

$$S = 2\pi r h + 2\pi r^2 \text{ cm}^2,$$

since it consists of the lateral side and two circular ends and is the **objective function**. The condition on the volume gives the height

$$h = \frac{1000}{\pi r^2}.$$

Substituting this expression into the equation for the surface area, we obtain our optimization problem in the one variable r:

$$S(r) = 2\pi r \frac{1000}{\pi r^2} + 2\pi r^2 = 2000r^{-1} + 2\pi r^2.$$

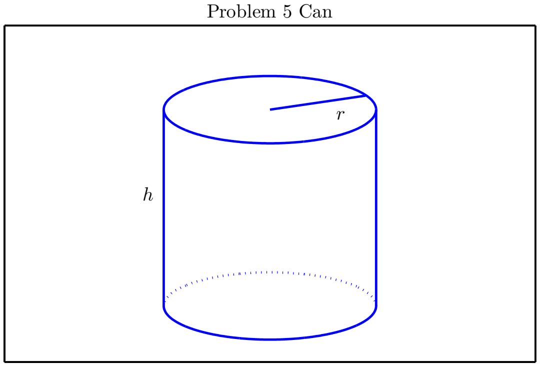
To find the **optimal solution** we differentiate and set the expression equal to zero:

$$S'(r) = -2000r^{-2} + 4\pi r = \frac{4\pi r^3 - 2000}{r^2} = 0.$$

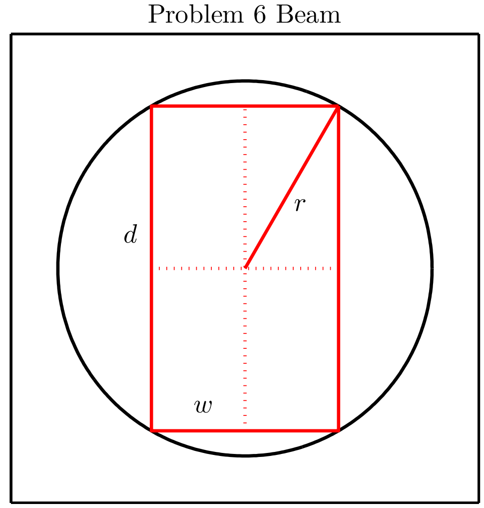
Thus, $4\pi r^3 - 2000 = 0$ or $r_{min} = \frac{10}{\sqrt[3]{2\pi}} \approx 5.419$ cm. It easily follows that $h_{min} = \frac{1000}{\pi}\left(\frac{\sqrt[3]{2\pi}}{10}\right)^2 = 10\sqrt[3]{\frac{4}{\pi}} \approx 10.839$ cm. The least amount of construction material for this one liter can is

$$S(r_{min}) = 200\sqrt[3]{2\pi} + \frac{200\pi}{\sqrt[3]{4\pi^2}} = 300\sqrt[3]{2\pi} \approx 553.58 \text{ cm}^2.$$

Problem 5 Can



Problem 6 Beam



6. The beam with width w and depth d that is cut from a circular log with a radius of r is shown in the diagram above on the right. The **objective function** for maximizing strength satisfies:

$$S = kwd^2,$$

where the constant k for the proportionality constant. The **constraint condition** is

$$\left(\frac{w}{2}\right)^2 + \left(\frac{d}{2}\right)^2 = r^2 \quad \text{or} \quad w^2 + d^2 = 4r^2 \quad \text{or} \quad d^2 = 4r^2 - w^2.$$

The **optimization problem** with $r = 25$ has $d^2 = 2500 - w^2$ and becomes

$$S(w) = kw(2500 - w^2) = k(2500w - w^3).$$

To find the strongest beam, we differentiate $S(w)$, then set the derivative equal to zero:

$$S'(w) = k(2500 - 3w^2) = 0 \quad \text{or} \quad 3w^2 = 2500 \quad \text{or} \quad w_{max} = \frac{50}{\sqrt{3}} \approx 28.8675 \text{ cm}.$$

The optimal depth satisfies

$$d_{max}^2 = 2500 - \frac{2500}{3} \quad \text{or} \quad d_{max} = 50\sqrt{\frac{2}{3}} \approx 40.8248 \text{ cm}.$$

7. The population density as a function of nutrient c satisfies $P(c) = \frac{1000c}{1+100c^2}$, which is the **objective function**. The maximum value of $P(c)$ occurs when the derivative of P with respect to c is zero. We use the quotient rule, so

$$P'(c) = 1000\frac{(1 + 100c^2) - c(200c)}{(1 + 100c^2)^2} = \frac{1000(1 - 100c^2)}{(1 + 100c^2)^2}.$$

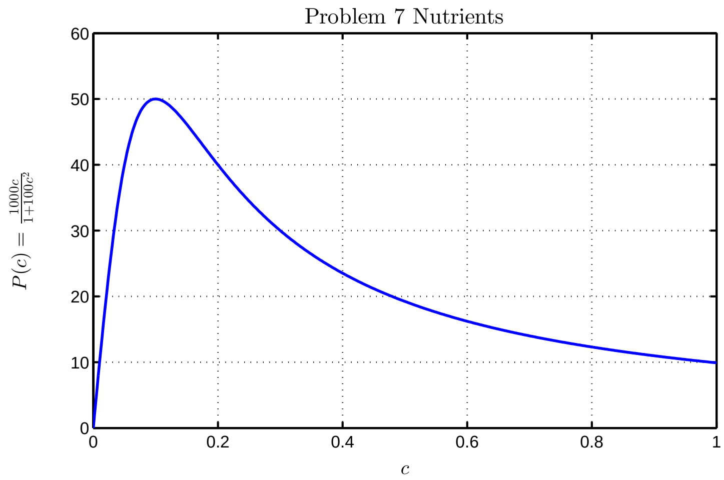
This is zero when the numerator is zero, so

$$1 - 100c^2 = 0 \qquad \text{or} \qquad c_m = 0.1.$$

Since

$$P(0.1) = \frac{1000(0.1)}{1 + 100(0.1)^2} = 50,$$

we have that the optimal concentration is $c_m = 0.1$ M, and the maximal population density is $P_{max} = 50$ organisms/cm^2. Note the graph passes through the origin and has a horizontal asymptote of $P = 0$ for large values of c (since the power of the denominator exceeds the power of the numerator). The graph of this function is below.



8. The profitability function satisfies $P(x) = Kx\frac{(r-x)}{r} = K\left(x - \frac{x^2}{r}\right)$, where x is the level of harvesting. The maximum profit occurs when $P'(x) = K\left(1 - \frac{2x}{r}\right) = 0$. This happens when $x_{max} = \frac{r}{2}$, which is the vertex of the quadratic profit function with $P(r/2) = \frac{Kr}{4}$. The equilibrium fish population at this level of harvesting is $F_e = K\frac{(r-\frac{r}{2})}{r} = \frac{K}{2}$ fish.

The equilibrium population of fish is $F_e(x) = \frac{K}{r}(r-x)$ with $F_e'(x) = -\frac{K}{r} < 0$. This is a decreasing function of x, so the maximum number of fish occurs when harvesting effort $x = 0$, with equilibrium population of fish at K, the carrying capacity.

9. The model for age-structured populations is given by

$$r(x) = \frac{\ln(e^{-0.21x}6x^{1.1})}{x} = \frac{\ln(e^{-0.21x}) + \ln(6) + 1.1\ln(x)}{x} = -0.21 + \ln(6)x^{-1} + 1.1\frac{\ln(x)}{x}.$$

The derivative is given by

$$r'(x) = -\ln(6)x^{-2} + 1.1\frac{x(\frac{1}{x}) - \ln(x)}{x^2} = \frac{1.1 - \ln(6) - 1.1\ln(x)}{x^2}.$$

The optimal rate of increase is assumed to be the maximum rate. Setting the derivative equal to zero, we take only the numerator from the derivative above. Thus,

$$1.1 - \ln(6) - 1.1\ln(x) = 0 \qquad \text{or} \qquad \ln(x) = 1 - \frac{\ln(6)}{1.1}.$$

Exponentiating

$$x_{opt} = e^{\left(1-\frac{\ln(6)}{1.1}\right)} = e \cdot 6^{-1/1.1} \approx 0.53319.$$

Thus, the optimal age of reproduction is $x_{opt} = e \cdot 6^{-1/1.1} \approx 0.53319$.

10. a. From the diagram, $d_1 = 500 - x$ and $d_2 = \sqrt{x^2 + 200^2}$ therefore

$$T(x) = \frac{500-x}{10} + \frac{\sqrt{x^2+200^2}}{6} = 50 - \frac{x}{10} + \frac{1}{6}\left(x^2 + 200^2\right)^{\frac{1}{2}}.$$

b. To find the minimal time to the den, we differentiate $T(x)$ and find when $T'(x) = 0$. The derivative satisfies:

$$T'(x) = -\frac{1}{10} + \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)\left(x^2 + 200^2\right)^{-\frac{1}{2}} 2x = -\frac{1}{10} + \left(\frac{x}{6}\right)\left(x^2 + 200^2\right)^{-\frac{1}{2}} = 0.$$

We solve this for x.

$$\left(x^2 + 200^2\right)^{\frac{1}{2}} = \frac{10x}{6} \qquad \text{or} \qquad x^2 + 200^2 = \frac{25}{9}x^2.$$

Thus,

$$\frac{16}{9}x^2 = 200^2 \qquad \text{or} \qquad x = 200\left(\frac{3}{4}\right) = 150.$$

The minimum distance is $x_{min} = 150$ ft, so the otter runs 350 ft along the shore, then goes straight to the den, which is 250 ft through the water. The minimum time is

$$T(x_{min}) = \frac{500 - 150}{10} + \frac{\sqrt{150^2 + 200^2}}{6} \approx 78.667 \text{ sec.}$$

11. a. The intensity of the lamp at the edge of the table satisfies, $I = \frac{3.4\cos(\theta)}{d^2}$. From the diagram we can write $d = \frac{4}{\sin(\theta)}$. It follows that

$$I(\theta) = 3.4\cos(\theta)\left(\frac{\sin(\theta)}{4}\right)^2 = \frac{3.4}{16}\cos(\theta)\sin^2(\theta).$$

The derivative (product and chain rules) satisfies

$$I'(\theta) = \frac{3.4}{16}\left(2\sin(\theta)\cos^2(\theta) - \sin^3(\theta)\right) = \frac{3.4}{16}\sin(\theta)\left(2\cos^2(\theta) - \sin^2(\theta)\right).$$

b. To maximize the illumination at the edge, $I'(\theta) = 0$, either $\sin(\theta) = 0$ or $2\cos^2(\theta) = \sin^2(\theta)$. If $\sin(\theta) = 0$, then $\theta = 0$ and the lamp has $h = \infty$, which is a minimum. Using $\sin^2(\theta) + \cos^2(\theta) = 1$ or $\sin^2(\theta) = 1 - \cos^2(\theta)$, we have

$$2\cos^2(\theta) = 1 - \cos^2(\theta) \quad \text{or} \quad 3\cos^2(\theta) = 1 \quad \text{or} \quad \cos(\theta) = \frac{1}{\sqrt{3}} \quad \text{or} \quad \theta_c = \arccos\left(\frac{1}{\sqrt{3}}\right).$$

The basic definitions of trigonometric functions give $\tan(\theta_c) = \sqrt{2}$. However, $\tan(\theta_c) = \frac{4}{h} = \sqrt{2}$, so $h = \frac{4}{\sqrt{2}} \approx 2.8284$ ft.

12. a. The surface area of the beehive cells satisfies:

$$S(\theta) = \frac{4V\sqrt{3}}{3R} - \frac{3R^2}{2}\frac{\cos(\theta)}{\sin(\theta)} + \frac{3R^2\sqrt{3}}{2}\frac{1}{\sin(\theta)}.$$

The derivative of $S(\theta)$ is:

$$\begin{aligned} S'(\theta) &= 0 - \frac{3R^2}{2}\left(\frac{-\sin(\theta)\cdot\sin(\theta) - \cos(\theta)\cdot\cos(\theta)}{\sin^2(\theta)}\right) + \frac{3R^2\sqrt{3}}{2}(-1)\left(\frac{\cos(\theta)}{\sin^2(\theta)}\right) \\ &= \frac{3R^2}{2\sin^2(\theta)}\left(1 - \sqrt{3}\cos(\theta)\right). \end{aligned}$$

b. The minimum surface area occurs where $S'(\theta) = 0$, therefore $1 - \sqrt{3}\cos(\theta) = 0$ or

$$\cos(\theta) = \frac{1}{\sqrt{3}}.$$