1. The initial value problem, $\frac{dy}{dt} = 0.3y$ with $y(0) = 20$, has the solution:

$$y = 20e^{0.3t}.$$

The exact value of this solution at $t = 1$ is $y(1) = 20e^{0.3} \approx 26.9972$.

It requires five steps to use Euler's method to approximate the solution using a stepsize of $h = 0.2$ for $t \in [0, 1]$. The Euler formula for this problem is

$$y_{n+1} = y_n + 0.2(0.3y_n) = y_n + 0.06y_n.$$

Below is a table showing the iterations for Euler's solution.

t_n	y_n
$t_0 = 0$	$y_0 = 20$
$t_1 = 0.2$	$y_1 = y_0 + 0.06y_0 = 20 + 0.06(20) = 21.2$
$t_2 = 0.4$	$y_2 = y_1 + 0.06y_1 = 21.2 + 0.06(21.2) = 22.472$
$t_3 = 0.6$	$y_3 = y_2 + 0.06y_2 = 22.472 + 0.06(22.472) = 23.8203$
$t_4 = 0.8$	$y_4 = y_3 + 0.06y_3 = 23.8203 + 0.06(23.8203) = 25.2495$
$t_5 = 1.0$	$y_5 = y_4 + 0.06y_4 = 25.2495 + 0.06(25.2495) = 26.7645$

The percent error is $100\left(\frac{26.7645-26.9972}{26.9972}\right) \approx -0.862\%$.

2. Consider the initial value problem, $\frac{dy}{dt} = 10 - 0.3y = -0.3\left(y - \frac{100}{3}\right)$ with $y(0) = 10$. Make the substitution $z(t) = y(t) - \frac{100}{3}$, so $z(0) = y(0) - \frac{100}{3} = -\frac{70}{3}$. The translated problem becomes:

$$\frac{dz}{dt} = -0.3z, \qquad \text{with} \qquad z(0) = -\frac{70}{3},$$

which has the solution

$$z(t) = -\frac{70}{3}e^{-0.3t} \qquad \text{or} \qquad y(t) = \frac{100}{3} - \frac{70}{3}e^{-0.3t}.$$

The exact value of this solution at $t = 1$ is $y(1) = \frac{100}{3} - \left(\frac{70}{3}\right)e^{-0.3} \approx 16.0476$.

Euler's method with stepsize of $h = 0.2$ for $t \in [0, 1]$ gives the following formula for this problem:

$$y_{n+1} = y_n + 0.2(10 - 0.3y_n) = 0.94y_n + 2.$$

Below is a table showing the iterations for Euler's solution.

t_n	y_n
$t_0 = 0$	$y_0 = 10$
$t_1 = 0.2$	$y_1 = 0.94y_0 + 2 = 9.4 + 2 = 11.4$
$t_2 = 0.4$	$y_2 = 0.94y_1 + 2 = 10.716 + 2 = 12.716$
$t_3 = 0.6$	$y_3 = 0.94y_2 + 2 = 11.953 + 2 = 13.953$
$t_4 = 0.8$	$y_4 = 0.94y_3 + 2 = 13.116 + 2 = 15.116$
$t_5 = 1.0$	$y_5 = 0.94y_4 + 2 = 14.2089 + 2 = 16.2089$

The percent error is $100\left(\frac{16.2089-16.0476}{16.0476}\right) \approx 1.005\%$.

3. a. The population model satisfies $\frac{dP}{dt} = 0.25P - 15 = 0.25\,(P-60)$ with $P(0) = 320$. Make the substitution $z(t) = P(t) - 60$, so $z(0) = P(0) - 60 = 260$. The translated problem becomes:

$$\frac{dz}{dt} = 0.25z, \qquad \text{with} \qquad z(0) = 260,$$

which has the solution

$$z(t) = 260e^{0.25t} \qquad \text{or} \qquad P(t) = 60 + 260e^{0.25t}.$$

The exact value of this solution at $t = 1$ is $P(1) = 60 + 260e^{0.25} \approx 393.85$.

b. Euler's method with stepsize of $h = 0.25$ for $t \in [0,1]$ gives the following formula for this problem:

$$P_{n+1} = P_n + 0.25(0.25P_n - 15) = 1.0625P_n - 3.75.$$

Below is a table showing the iterations for Euler's solution.

t_n	P_n
$t_0 = 0$	$P_0 = 320$
$t_1 = 0.25$	$P_1 = 1.0625P_0 - 3.75 = 336.25$
$t_2 = 0.5$	$P_2 = 1.0625P_1 - 3.75 = 353.5156$
$t_3 = 0.75$	$P_2 = 1.0625P_2 - 3.75 = 371.8604$
$t_4 = 1.0$	$P_4 = 1.0625P_3 - 3.75 = 391.3516$

The percent error is $100\left(\frac{391.35-393.85}{393.85}\right) \approx -0.633\%$.

4. a. The body temperature satisfies $\frac{dT}{dt} = -0.2(T-22)$ with $T(0) = 36$. Make the substitution $z(t) = T(t) - 22$, so $z(0) = T(0) - 22 = 14$. The translated problem becomes:

$$\frac{dz}{dt} = -0.2z, \qquad \text{with} \qquad z(0) = 14,$$

which has the solution

$$z(t) = 14e^{-0.2t} \qquad \text{or} \qquad T(t) = 22 + 14e^{-0.2t}.$$

The exact value of this solution at $t = 2$ is $T(2) = 22 + 14e^{-0.4} \approx 31.384$.

b. Euler's method with stepsize of $h = 0.5$ for $t \in [0,2]$ gives the following formula for this problem:

$$T_{n+1} = T_n + 0.5(-0.2(T_n - 22)) = 0.9T_n + 2.2.$$

Below is a table showing the iterations for Euler's solution.

t_n	T_n
$t_0 = 0$	$T_0 = 36$
$t_1 = 0.5$	$T_1 = 0.9T_0 + 2.2 = 32.4 + 2.2 = 34.6$
$t_2 = 1.0$	$T_2 = 0.9T_1 + 2.2 = 31.14 + 2.2 = 33.34$
$t_3 = 1.5$	$T_3 = 0.9T_2 + 2.2 = 30.006 + 2.2 = 32.206$
$t_4 = 2.0$	$T_4 = 0.9T_3 + 2.2 = 28.985 + 2.2 = 31.185$

The percent error is $100\left(\frac{31.185-31.384}{31.384}\right) \approx -0.634\%$.

c. The new differential equation is

$$\frac{dT}{dt} = -0.2(T - (22 - 0.5t)).$$

Euler's method with stepsize of $h = 0.5$ for $t \in [0, 2]$ gives the following formula for this problem:

$$T_{n+1} = T_n + 0.5(-0.2(T_n - (22 - 0.5t))) = 0.9T_n + 0.1(22 - 0.5t).$$

Below is a table showing the iterations for Euler's solution.

t_n	T_n
$t_0 = 0$	$T_0 = 36$
$t_1 = 0.5$	$T_1 = 0.9T_0 + 0.1(22 - 0.5(0)) = 34.6$
$t_2 = 1.0$	$T_2 = 0.9T_1 + 0.1(22 - 0.5(0.5)) = 33.315$
$t_3 = 1.5$	$T_3 = 0.9T_2 + 0.1(22 - 0.5(1.0)) = 32.1335$
$t_4 = 2.0$	$T_4 = 0.9T_3 + 0.1(22 - 0.5(1.5)) = 31.045$

5. a. The differential equation describing the radioactive decay is $\frac{dR}{dt} = -0.25R + 7e^{-0.05t}$ with $R(0) = 70$. The Euler's solution with a stepsize of $h = 0.5$ for $t \in [0, 3]$. The Euler formula for this problem is

$$\begin{aligned} R_{n+1} &= R_n + 0.5\left(-0.25R_n + 7e^{-0.05t_n}\right), \\ R_{n+1} &= 0.875R_n + 3.5e^{-0.05t_n}. \end{aligned}$$

Below is a table showing the iterations for this Euler's solution.

t_n	R_n
$t_0 = 0$	$R_0 = 70$
$t_1 = 0.5$	$R_1 = 0.875R_0 + 7e^{-0.05t_0} = 0.875(70) + 7 = 64.75$
$t_2 = 1.0$	$R_2 = 0.875R_1 + 7e^{-0.05t_1} = 60.07$
$t_3 = 1.5$	$R_3 = 0.875R_2 + 7e^{-0.05t_2} = 55.89$
$t_4 = 2.0$	$R_4 = 0.875R_3 + 7e^{-0.05t_3} = 52.15$
$t_5 = 2.5$	$R_5 = 0.875R_4 + 7e^{-0.05t_4} = 48.80$
$t_6 = 3.0$	$R_6 = 0.875R_5 + 7e^{-0.05t_5} = 45.79$

b. To verify that $R(t) = 35e^{-0.05t} + 35e^{-0.25t}$ is a solution to the initial value problem, $R(0) = 35 \cdot 1 + 35 \cdot 1 = 70$. Also we differentiate the proposed solution, giving

$$R' = -0.05\left(35e^{-0.05t}\right) - 0.25\left(35e^{-0.25t}\right) = -1.75e^{-0.05t} - 8.75e^{-0.02t}.$$

For the right hand side of the equation, we substitute the solution giving

$$\begin{aligned} -0.25R + 7e^{-0.05t} &= -0.25\left(35e^{-0.05t} + 35e^{-0.25t}\right) + 7e^{-0.05t} \\ &= -1.75e^{-0.05t} - 8.75e^{-0.02t}. \end{aligned}$$

Thus, the equation is satisfied. The solution at $t = 3$ is $R(3) = 35e^{-0.05 \cdot 3} + 35e^{-0.25 \cdot 3} = 46.6576$. The percent error is therefore $100\left(\frac{45.7881-46.6576}{46.6576}\right) = -1.864\%$.