1. Consider the function $f(x) = (x^2 - 3x + 4)^4$. This is a composite of $h(u) = u^4$ and $u(x) = x^2 - 3x + 4$. Since $h'(u) = 4u^3$, the chain rule gives

$$f'(x) = h'(u(x))u'(x) = 4(x^2 - 3x + 4)^3(2x - 3).$$

2. The function $f(x) = \ln(22 + \sin(7x))$ is the composite of two functions $g(u) = \ln(u)$ and $u(x) = 22 + \sin(7x)$.

$$f'(x) = g'(u(x))u'(x) = \frac{1}{(22 + \sin(7x))} \cdot (0 + 7\cos(7x)) = \frac{7\cos(7x)}{22 + \sin(7x)}.$$

3. Consider the function $f(x) = e^{-6x}(3x + 15)^2 + \ln(x^5 - 10)$. Use the product rule for the first term, $x$, and then chain rule.

$$\begin{aligned} f'(x) &= e^{-6x}(2)(3x+15)^1(3) - 6e^{-6x}(3x+15)^2 + \frac{1}{(x^5 - 10)}5x^4 \\ &= -6e^{-6x}(3x+15)(3x+14) + \frac{5x^4}{x^5 - 10}. \end{aligned}$$

4. Consider the function $f(x) = x\sin(x^2 - 7)$. Differentiation uses the product with the second factor requiring the chain rule for its derivative:

$$f'(x) = x\cos(x^2 - 7)(2x) + 1 \cdot \sin(x^2 - 7) = 2x^2\cos(x^2 - 7) + \sin(x^2 - 7).$$

5. Consider $f(x) = \left(x^2 - e^{-x^3}\right)^{-1} + \frac{1}{x^2+4} = \left(x^2 - e^{-x^3}\right)^{-1} + (x^2 + 4)^{-1}$. Application of the chain rule gives:

$$\begin{aligned} f'(x) &= -1\left(x^2 - e^{-x^3}\right)^{-2}(2x - e^{-x^3}(-3x^2)) - 1(x^2+4)^{-2}(2x) \\ &= \frac{-(2x + 3x^2e^{-x^3})}{(x^2 - e^{-x^3})^2} - \frac{2x}{(x^2+4)^2}. \end{aligned}$$

6. Consider the function $f(x) = (x^4 - \cos(4x^5))^6$. From the chain rule the derivative is

$$\begin{aligned} f'(x) &= 6\left(x^4 - \cos(4x^5)\right)^5(4x^3 + \sin(4x^5)(4 \cdot 5x^4)) \\ &= 24x^3(1 + 5x\sin(4x^5))\left(x^4 - \cos(4x^5)\right)^5. \end{aligned}$$

7. Consider the function $f(x) = \frac{1}{\sin^2(x^3)} = (\sin(x^3))^{-2}$. Rewrite the function,

$$f'(x) = -2(\sin(x^3))^{-3}\cos(x^3) \cdot 3x^2 = -\frac{6x^2\cos x^3}{\sin^3(x^3)}.$$

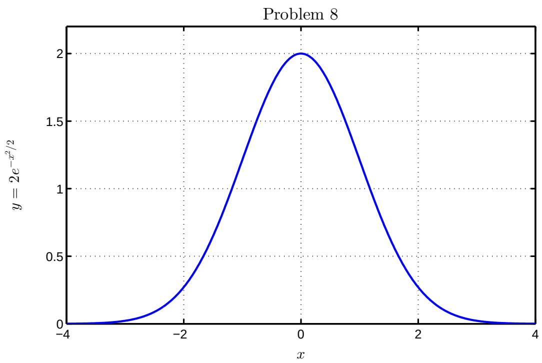
8. Consider the function $y = 2e^{-\frac{x^2}{2}}$. By the chain rule the derivative is

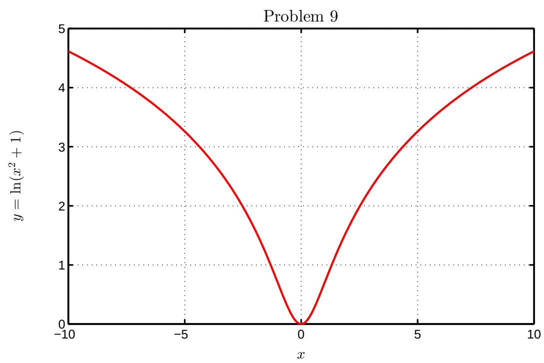
$$y' = 2e^{-\frac{x^2}{2}}\left(\frac{-2x}{2}\right) = -2xe^{-\frac{x^2}{2}}.$$

The product rule followed by the chain rule gives the second derivative as

$$y'' = -2x \cdot e^{-\frac{x^2}{2}}(-x) - 2e^{-\frac{x^2}{2}} = 2(x^2 - 1)e^{-\frac{x^2}{2}}.$$

The $y$-intercept satisfies $x = 0$ or $y(0) = 2e^0 = 2$. The $x$-intercept would satisfy $y = 0 = 2e^{-x^2/2}$, which does not exist, since the exponential function is always positive. Since the exponential function decays very rapidly as $x \to \pm\infty$, there is a horizontal asymptote with $y = 0$. The critical points are where $y' = 0$, or when $x_c = 0$. This is the $y$-intercept $(0, 2)$. The points of inflection occur where $y'' = 0$, so $(x^2 - 1) = 0$. Thus, $x_{1p} = -1$ with $y(x_{1p}) = 2e^{-\frac{1}{2}} \approx 1.21306$. Similarly, $x_{2p} = 1$ with $y(x_{2p}) = 2e^{-\frac{1}{2}} \approx 1.21306$. This is an even function, reflected across the $y$-axis. The graph is shown below to the left.




9. The function $y = \ln(x^2+1)$, is the composite of two functions $g(u) = \ln(u)$ and $u(x) = x^2+1$. The derivative satisfies

$$y' = \frac{1}{x^2+1}(2x) = \frac{2x}{x^2+1} = 2x(x^2+1)^{-1}.$$

The second derivative uses the quotient rule giving

$$y'' = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{2(1-x^2)}{(x^2+1)^2}.$$

The $y$-intercept occurs when $y(0) = \ln(0+1) = 0$, giving the only intercept ($x$ and $y$) as $(0, 0)$. There are no asymptotes, since the domain is all $x$ (no vertical) and the logarithmic function is unbounded (no horizontal). Critical points occur when $y' = 0$, so the numerator $2x = 0$ or $x_c = 0$. Thus, there is a minimum at $(0, 0)$. There are points of inflection where $y'' = 0 = (1 - x^2)$ so $x_{1p} = -1$ and $y(x_{1p}) = \ln(2) \approx 0.69315$, with $x_{2p} = 1$ and $y(x_{2p}) = \ln(2)$. This is an even function. The graph is above to the right.

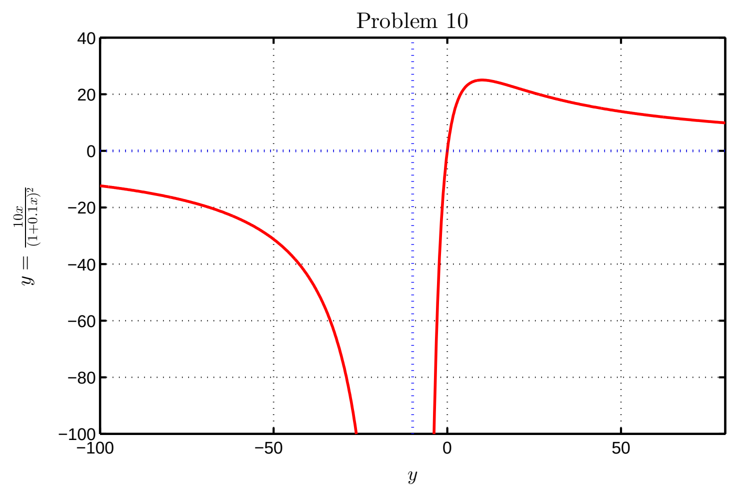
10. Consider the function $y = \frac{10x}{(1+0.1x)^2}$. We apply the quotient and chain rule to yield:

$$y' = 10\frac{(1+0.1x)^2 \cdot 1 - 2(1+0.1x)^1(0.1)x}{(1+0.1x)^4} = \frac{10-x}{(1+0.1x)^3} = (10-x)(1+0.1x)^{-3}.$$

The product rule and chain rule can be used for the second derivative:

$$\begin{aligned} y'' &= (10-x)(-3)(1+0.1x)^{-4}(0.1) - 1(1+0.1x)^{-3} \\ &= -(3(1-0.1x)+(1+0.1x))(1+0.1x)^{-4} = \frac{0.2(x-20)}{(1+0.1x)^4} \end{aligned}$$

The $y$-intercept satisfies $y(0)=0$, so the origin is both the $x$ and $y$-intercept. The domain is $x \neq -10$, which gives a vertical asymptote at $x=-10$. Since the leading power of the denominator exceeds the leading power of the numerator, there is a horizontal asymptote at $y=0$. At the critical points $y'=0$, so $x_c=10$ and $y(x_c)=\frac{10\cdot 10}{(1+0.1\cdot 10)^2}=25$, which is a maximum. The point of inflection is where $y''=0$, so $x_i=20$ and $y(x_i)=\frac{10\cdot 20}{(1+0.1\cdot 20)^2}=\frac{200}{9}\approx 22.22$. This function is neither even nor odd. The graph is below.




11. a. The height equation is $h(a)=6.44a+82.1$. It follows that the rate of growth in height is given by the derivative of the height equation, so the rate of growth is $h'(a)=6.44$ cm/yr.

b. Weight as a function of age $W(a)=0.0000302(6.44a+82.1)^{2.84}$. The derivative is

$$W'(a)=0.0000302(2.84)(6.44)(6.44a+82.1)^{1.84}=0.000552346(6.44a+82.1)^{1.84} \text{ kg/yr}.$$

c. The rate of change of weight at ages 4, 8, and 13 is found by substituting into the expression above for the derivative giving:

$$W'(4)=3.0386 \text{ kg/yr}, \quad W'(8)=4.5062 \text{ kg/yr}, \quad \text{and} \quad W'(13)=6.7041 \text{ kg/yr}.$$

12. a. Consider Hassell's model given by $H(P)=\frac{5P}{(1+0.002P)^4}$. The equilibria of the model solve the equation $H(P_e)=\frac{5P_e}{(1+0.002P_e)^4}=P_e$. Thus, we have the extinction equilibrium, $P_e=0$, or

$$5=(1+0.002P_e)^4 \quad \text{or} \quad P_e=500(5^{1/4}-1)\approx 247.67 \text{ (carrying capacity)}.$$

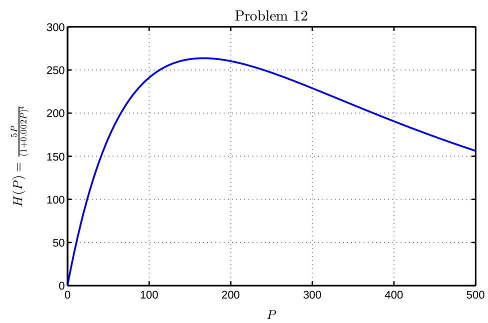
b. To differentiate this rate function $H'(P)$, we apply the quotient rule

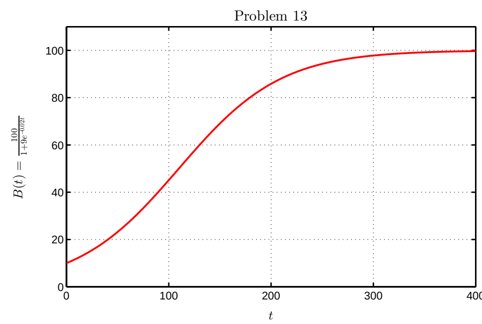
$$\begin{aligned} H'(P) &= 5\frac{(1+0.002P)^4\cdot 1 - P(4)(1+0.002P)^3(0.002)}{(1+0.002P)^8} \\ &= 5\frac{(1+0.002P)-0.008P}{(1+0.002P)^5} = 5\frac{(1-0.006P)}{(1+0.002P)^5} \end{aligned}$$

c. The $H$-intercept satisfies $H(0) = 0$. Thus, the origin, $(0, 0)$, is the $P$ and $H$-intercept. Since the power of the denominator exceeds the power of the numerator, there is a horizontal asymptote at $H = 0$. Because domain has $P \geq 0$, there is no vertical asymptote. The critical points satisfy $H'(P) = 0$, so $1 - 0.006P = 0$ or $P_c = \frac{500}{3} \approx 166.67$. It follows that

$$H(P_c) = 5\left(\frac{500}{3}\right)\left(1 + \frac{1}{3}\right)^{-4} \approx 263.67,$$

so there is a maximum at $(166.67, 263.67)$. The graph is shown below on the left.




13. a. Consider $B(t) = \frac{100}{1+9e^{-0.02t}} = 100(1 + 9e^{-0.02t})^{-1}$. The chain rule gives the derivative

$$B'(t) = -100(1 + 9e^{-0.02t})^{-2}(-0.18e^{-0.02t}) = \frac{18e^{-0.02t}}{(1 + 9e^{-0.02t})^2}.$$

We note that $B'(t) > 0$, so this function is always increasing. The second derivative is found by combining the quotient and chain rules to the expression for $B'(t)$ giving

$$\begin{aligned} B''(t) &= \frac{(1 + 9e^{-0.02t})^2(-0.36e^{-0.02t}) - 18e^{-0.02t}(2(1 + 9e^{-0.02t})(-0.18e^{-0.02t}))}{(1 + 9e^{-0.02t})^4} \\ &= \frac{0.36(9e^{-0.04t} - e^{-0.02t})}{(1 + 9e^{-0.02t})^3} = \frac{0.36e^{-0.02t}(9e^{-0.02t} - 1)}{(1 + 9e^{-0.02t})^3}. \end{aligned}$$

The point of inflection is found by solving $9e^{-0.02t} - 1 = 0$ or $e^{0.02t} = 9$. It follows that $t = 50\ln(9) \approx 109.86$ min and $B(109.86) = \frac{100}{1+9e^{-0.02 \cdot 109.86}} = 50.0$. Thus, the point of inflection is $(109.86, 50.0)$. The most rapid rate of growth is then

$$B'(109.86) = \frac{18e^{-0.02 \cdot 109.86}}{(1 + 9e^{-0.02 \cdot 109.86})^2} = 0.500 \text{ thousands of bacteria/ml/min.}$$

b. Since $B(0) = 100 = (1 + 9) = 10$, the $B$-intercept is $(0, 10)$. Since the exponential function tends to zero as $t \to +\infty$, the population $B(t)$ tends to 100. Thus, there is a horizontal asymptote at $B = 100$. The graph is shown above on the right.

14. a. The length of a sculpin satisfies $L(t) = 16\left(1 - e^{-0.4t}\right)$. Since $L(0) = 0$, the $t$ and $L$-intercept is $(0, 0)$. Since $e^{-0.4t} \to 0$ as $t \to +\infty$, there is a horizontal asymptote at $L = 16$ cm. The graph is shown below on the left.

b. Since $W(L) = 0.07L^3$, the composite function is given by

$$W(t) = 0.07(16)^3(1 - e^{-0.4t})^3 = 286.72(1 - e^{-0.4t})^3.$$

By similar arguments to Part a, this has a $t$ and $W$-intercept at $(0, 0)$, and the horizontal asymptote is $W = 286.72$ g. The graph is shown below on the right.

c. By the chain rule, the derivative of $W(t)$ is

$$W'(t) = 286.72(3(1 - e^{-0.4t})^2(0.4)e^{-0.4t}) = 344.064e^{-0.4t}(1 - e^{-0.4t})^2.$$

With the product and chain rules, the second derivative of $W(t)$ is

$$\begin{aligned} W''(t) &= 344.064(e^{-0.4t}2(1 - e^{-0.4t})(0.4)e^{-0.4t} - 0.4e^{-0.4t}(1 - e^{-0.4t})^2) \\ &= 137.6256e^{-0.4t}(1 - e^{-0.4t})(3e^{-0.4t} - 1). \end{aligned}$$

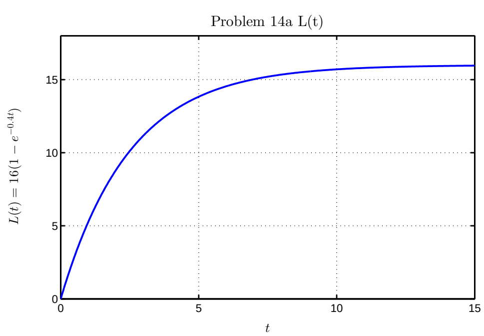
The weight of the sculpin is increasing most rapidly at the point of inflection, which occurs when

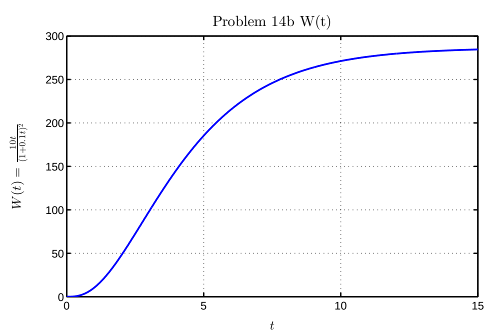
$$W''(t) = 0 \qquad \text{or} \qquad (1 - e^{-0.4t})(3e^{-0.4t} - 1) = 0.$$

The first factor is zero when $t = 0$ (which is when $W'(t) = 0$, so no weight increase), while the second factor is zero when

$$e^{0.4t} = 3 \qquad \text{or} \qquad t_i = \frac{5}{2}\ln(3) \approx 2.7465 \text{ yrs.}$$

The average sculpin weighs $W(t_i) = 84.954$ g at this age and is increasing in weight at a rate of $W'(t_i) = 50.972$ g/yr.




15. a. Consider the biomass model, $P(t) = 20\,(1 - e^{-0.2t})$. The $t$ and $P$-intercept for $P(t)$ is $(0, 0)$. Since the exponential goes to zero as $t \to \infty$, the horizontal asymptote is $P = 20$ metric tons. A graph of this function is shown below. The derivative (rate of change of biomass) of this function is

$$P'(t) = -20(-0.2)e^{-0.2t} = 4e^{-0.2t} \text{ metric tons/yr.}$$

The rate of change of biomass at $t = 0$ is $P'(0) = 4.0$ metric tons/yr.
At $t = 2$, $P'(2) = 2.6813$ metric tons/yr.
At $t = 10$, $P'(10) = 0.5413$ metric tons/yr.
At $t = 20$, $P'(20) = 0.07326$ metric tons/yr.

b. The composite function is given by

$$H(t) = H(P(t)) = 3\left(1 - e^{-2(1-e^{-0.2t})}\right).$$

The derivative satisfies

$$H'(t) = \frac{dH}{dP}\frac{dP}{dt} = -0.3e^{-0.1P}(4e^{-0.2t}) = 1.2e^{-2(1-e^{-0.2t})}e^{-0.2t}.$$

The rate of change of biomass of the herbivores at $t = 0$ is $H'(t) = 1.2$ metric tons/yr.
At $t = 2$, $H'(2) = 0.4160$ metric tons/yr.
At $t = 10$, $H'(10) = 0.02881$ metric tons/yr.
At $t = 20$, $H'(20) = 0.003085$ metric tons/yr.