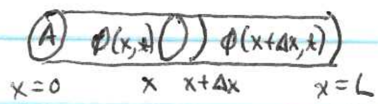
1.2 Heat Conduction in One-Dim. Rod.

Consider a rod of constant cross-sectional area A oriented in x-direction $x = 0$ to $x = L$

Define $e(x,t) \equiv$ thermal energy density

Assume lateral surface perfectly insulated, so thermal energy flows only in x-direction (rod not uniformly heated)



Heat energy - Consider a thin slice

$$\text{heat energy} = e(x,t)\underbrace{A\,\Delta x}_{\text{volume}} \quad \text{if } \Delta x \text{ suff. small}$$

Conservation of heat energy

rate of change of heat energy/time = heat energy flowing across boundaries/time + heat energy generated inside/time

$$\frac{\partial}{\partial t}\left[e(x,t)\,A\,\Delta x\right] = \text{rate of change of heat energy}$$

Heat flux $\phi(x,t)$ = heat flux (amt of thermal energy/time flowing to the right/surface area)

Heat sources $Q(x,t)$ = heat energy/volume/time

$$\therefore \frac{\partial}{\partial t}\left[e(x,t)\,A\,\Delta x\right] \approx \phi(x,t)A - \phi(x+\Delta x,t)A + Q(x,t)\,A\,\Delta x$$

$$\frac{\partial e}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x} + Q(x,t)$$

$$\therefore \frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q$$

Book has alternate exact derivation with integrals, but requires advanced calculus ideas for interchanging limiting processes

Temperature & specific heat

Let $u(x,t)$ = temperature

c = specific heat (or heat capacity) (uniform material ⇒ c constant)

$\rho(x)$ = mass density / volume

Relation of thermal energy & temperature

$$e(x,t) = c(x)\,\rho(x)\,u(x,t)$$

Fourier's Law $\phi = -K_0 \frac{\partial u}{\partial x}$ heat flows from hotter to colder regions

experimentaly shown proportional to temperature difference

K_0 = thermal conductivity K_0 small ⇒ poor conductance (insulation)

Heat Eqn.

$$c(x)\,\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + Q$$

For a uniform rod with no sources (or sinks) of heat

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad k = \frac{K_0}{c\rho} \quad \text{thermal diffusivity}$$

The term $\cancel{\phi}$ on r.h.s. of equation is thermal diffusion

Need I.C. $u(x,0) = f(x)$ & B.C. at $x=0$ & $x=L$.

Diffusion of a chemical pollutant (or other)

Let $u(x,t)$ be the concentration of a chemical

Consider a 1-D domain

Amt of chemical is

$$A\int_a^b u(x,t)\,dx$$

Let $\phi(x,t)$ be flux

Conservation

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \phi(a,t) - \phi(b,t)$$

No sources or sinks

$$\phi(a,t) - \phi(b,t) = -\int_a^b \frac{\partial \phi}{\partial x}\,dx$$

Fund. Th. of Calc.

$$\therefore \quad \frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = 0$$

Fick's Law of diffusion $\quad \phi = -k\frac{\partial u}{\partial x}$

$$\therefore \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

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1.3 Boundary Conditions

The heat eqn. needs information on the boundaries (along with I.C.'s)

<u>Prescribed temperature</u> or <u>Dirichlet</u> B.C.

$$u(0,t) = u_B(t)$$

$\nwarrow$ environmental temp.

<u>Insulated boundary</u> or <u>Neumann B.C.</u>

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = \phi(t) \leftarrow \text{prescribed heat flow}$$

$$\phi(t) = 0 \quad \text{for insulated}$$

<u>Newton's Law of cooling or Mixed B.C.</u> Robin

$$-K_0(0)\frac{\partial u}{\partial x}(0,t) = -H[u(0,t) - u_B(t)]$$

B.C.'s are Linear!

1.4 Equil. Temp. Distribution

Sample P.D.E.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$u(x,0) = f(x)$ $\quad u(0,t) = T_1(t)$, $u(L,t) = T_2(t)$

Equil. Spse $T_1(t) = T_1$ & $T_2(t) = T_2$ constant temp.

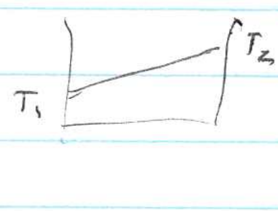
Equil or steady-state $\Rightarrow$ $\frac{\partial u}{\partial t} = 0$ and $u(x,t) = u(x)$

$\therefore$ Prob. above is

$$\frac{d^2 u}{dx^2} = 0 \qquad u(0) = T_1 \;,\; u(L) = T_2$$

$u(x) = c_1 + c_2 x$ $\qquad c_1 = T_1 \qquad c_2 = \frac{T_2 - T_1}{L}$

$\therefore$ $u(x) = T_1 + \frac{T_2 - T_1}{L} x$



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Insulated

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad u(x,0) = f(x)$$

$$\frac{\partial u(0,t)}{\partial x} = 0 = \frac{\partial u(L,t)}{\partial x}$$

Equil $\quad \frac{d^2 u}{dx^2} = 0 \qquad \frac{du}{dx}(0) = 0 = \frac{du}{dx}(L)$

Again $u(x) = c_1 x + c_2$ $\qquad$ B.C. $\Rightarrow c_1 = 0$

$\therefore$ $u(x) = c_2$

indeterminate from B.C.'s

$\lim_{t \to \infty} u(x,t) = C_2 \Rightarrow$ approaches const. temp.
physically reasonable

Since both ends are insulated, thermal energy is conserved

$$\frac{d}{dt} \int_0^L c\rho u \, dx = -K_0 \frac{\partial u}{\partial x}(0,t) + K_0 \frac{\partial u}{\partial x}(L,t) = 0$$

insulated

$$\therefore \int_0^L c\rho u \, dx = \text{constant}$$

Initial energy $\quad c\rho \int_0^L f(x)\, dx$

Same as $\quad c\rho \int_0^L u(x)\, dx = c\rho \int_0^L C_2 \, dx = c\rho C_2 L$

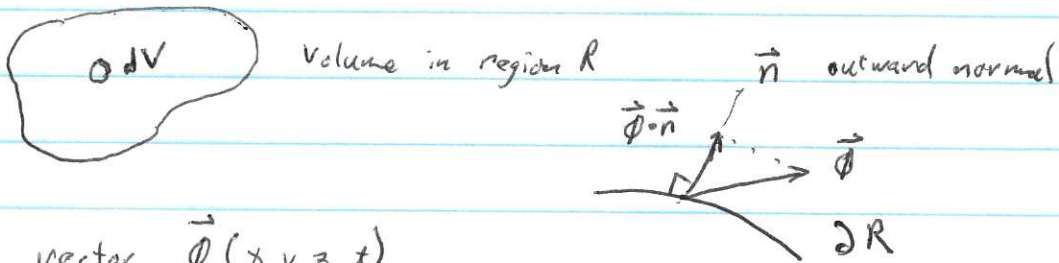
$$\therefore u(x) = C_2 = \frac{1}{L} \int_0^L f(x)\, dx$$

goes to average initial temp dist.

1.5 Derivation of Heat Eqn in 2D + 3D

Heat Energy

rate of change of heat energy = heat energy flowing across the boundaries per unit time + heat energy generated inside per unit time



Flux vector $\vec{\phi}(x, y, z, t)$

$$\text{Total energy} = \iiint_R c\rho u \, dV$$

Conservation of Heat energy

source/sink

$$\frac{d}{dt}\iiint_R c\rho u \, dV = -\oiint_{\partial R} \vec{\phi}\cdot\vec{n}\, ds + \iiint_R Q \, dV$$

Divergence Thm (Gauss's Thm) (Math 252) The volume integral of the divergence of any continuously differentiable vector A is the closed surface integral of the outward normal component of A:

$$\iiint_R \vec{\nabla}\cdot\vec{A}\, dV = \oiint_{\partial R} \vec{A}\cdot\vec{n}\, dS$$

Apply to Conservation of Heat Energy

$$\frac{d}{dt}\iiint_R c\rho u \, dV = -\iiint_R \vec{\nabla}\cdot\vec{\phi}\, dV + \iiint_R Q\, dV$$

$$\therefore \iiint_R \left[c\rho \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{\phi} - Q\right] dV = 0$$

Since this is zero for all regions R, it follows that

$$c\rho \frac{\partial u}{\partial t} = -\vec{\nabla}\cdot\vec{\phi} + Q$$

<u>Fourier's law of heat conduction</u> $\vec{\phi} = -K_0 \vec{\nabla} u$

$\left(\vec{\nabla} u \equiv \frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k} \right)$ gradient of temp (max heat change)

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<u>Heat equation:</u> $c\rho \frac{\partial u}{\partial t} = \vec{\nabla}\cdot\left(K_0 \vec{\nabla} u\right) + Q$.

When there are no heat sources or sinks $Q=0$ & thermal coefficients are const, then

$$\frac{\partial u}{\partial t} = k\,\vec{\nabla}\cdot\vec{\nabla} u = k\,\nabla^2 u$$

Note: $\vec{\nabla}\cdot\vec{\nabla} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

<u>Heat Problem:</u>

$$\frac{\partial u}{\partial t} = k\,\nabla^2 u \quad , \qquad \text{I.C. } u(x,y,z,0) = f(x,y,z)$$

<u>Prescribed</u> B.C. $u(x,y,z,t) = T(x,y,z,t)$ on ∂R boundary

known

<u>Insulated</u> $\vec{\nabla} u \cdot \vec{n} = 0$ on ∂R

<u>Mixed</u> $-K_0 \vec{\nabla} u \cdot \vec{n} = H(u - u_b)$ on ∂R

<u>Steady State</u> or <u>Poisson's Eqn:</u>

$$\nabla^2 u = -Q/K_0$$

if $Q=0$

$$\nabla^2 u = 0 \qquad \text{(Laplace's Eqn. or Potential Eqn.)}$$

Other coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Laplacian rectangular coor

Cylindrical Coord $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Spherical Coord $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$

$$\nabla^2 u = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

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Chap 2

Separation of Variables

General Heat Eqn. for rod $(0 \le x \le L)$

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho} \qquad \begin{matrix} t>0 \\ 0<x<L \end{matrix}$$

I.C. $u(x,0) = f(x)$, $0<x<L$

Dirichlet B.C. $u(0,t) = T_1(t)$, $u(L,t) = T_2(t)$ $\quad t>0$

If $Q(x,t) = 0$, PDE is homogeneous

If $T_1 = T_2 = 0$, B.C. are homogeneous

Usually split PDE into homogeneous + nonhomogeneous, then superimpose soln.

__Linearity__ (think subspaces from Linear Algebra)

An operator L is linear iff

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

for any two fcns u_1 + u_2 and constants c_1 + c_2

__Heat Operator__ $\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} = L$

$$L(c_1 u_1 + c_2 u_2) = \left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)(c_1 u_1 + c_2 u_2)$$

$$= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} - kc_2\frac{\partial^2 u_2}{\partial x^2} = c_1 L(u_1) + c_2 L(u_2)$$

$\therefore$ linear

Thus, Heat Eqn. is a linear PDE

__Principle of Superposition__ If u_1 + u_2 satisfy a linear homogeneous equation $(L(u) = 0)$, then any arbitrary linear combination, $c_1 u_1 + c_2 u_2$, also satisfies the linear homogeneous eqn.

<u>Note</u>: concepts of linearity & homogeneity also apply to B.C.'s

<u>Heat Eqn w/ Zero Temperature Finite Ends</u>

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad \begin{matrix} 0 < x < L \\ t > 0 \end{matrix} \qquad \text{I.C.} \quad u(x,0) = f(x)$$

B.C. $u(0,t) = u(L,t) = 0$

Linear homogeneous PDE w/ Linear homogeneous B.C.'s

<u>Note</u>: Equil soln. $u(x) \equiv 0$

<u>Sep. of Variables</u>

Consider $u(x,t) = \phi(x)\,G(t)$ Daniel Bernoulli 1700s

<u>Note</u>: Generally doesn't satisfy I.C., but ignore for now

Substitute into Heat Eqn.

$$\phi(x)\frac{dG}{dt} = k\,\frac{d^2\phi}{dx^2}\,G(t)$$

Separating variables

$$\frac{1}{G}\frac{dG}{dt} = k\,\frac{1}{\phi}\frac{d^2\phi}{dx^2} \iff \frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2}$$

l.h.s depends only on t & r.h.s. depends only on x

Independent variables $\Rightarrow$ = const.

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

$$\therefore \quad \frac{dG}{dt} = -\lambda k G \qquad \text{and} \qquad \frac{d^2\phi}{dx^2} = -\lambda\phi$$

<u>B.C.'s</u> $u(0,t) = G(t)\,\phi(0) = 0 \Rightarrow \phi(0) = 0$ (Don't want $G(t) \equiv 0$)

similarly $u(L,t) = G(t)\,\phi(L) = 0 \Rightarrow \phi(L) = 0$

Time dependent Eqn (ODE)

$$\frac{dG}{dt} = -\lambda k G$$

$$\therefore \; G(t) = c e^{-\lambda k t}$$

Note: $\lambda > 0 \Rightarrow$ decaying soln. $\quad \lambda = 0$ constant soln

$\lambda < 0 \Rightarrow$ growing soln. $\quad \therefore$ expect $\lambda \geq 0$, later show.

BVP $\quad \frac{d^2\phi}{dx^2} + \lambda\phi = 0, \quad \phi(0) = 0, \quad \phi(L) = 0$

ODE boundary value problem. Note $\phi(x) \equiv 0$ is a soln (trivial soln), but only satisfies trivial I.C.

Seek nontrivial solns.

$$L\phi = -\lambda\phi \quad \text{become eigenvalue Problem} \quad L = \frac{d^2}{dx^2}$$

with eigenvalues λ and eigenfcns ϕ.

4 cases to consider

1. $\lambda > 0$ 2. $\lambda = 0$ 3. $\lambda < 0$ 4. λ complex

(We'll ignore case 4 & later prove λ must be real.)

Case 2. $\lambda = 0$, then $\frac{d^2\phi}{dx^2} = 0, \quad \phi(0) = 0, \quad \phi(L) = 0$

$\phi(x) = c_1 x + c_2 \qquad \phi(0) = 0 \Rightarrow c_2 = 0 \qquad \phi(L) = 0 \Rightarrow c_1 = 0$

Thus, only trivial soln. (Not eigenfcn.)

Case 3 $\quad \lambda < 0 \; (\lambda = -\alpha^2)$

$\frac{d^2\phi}{dx^2} - \alpha^2\phi = 0 \qquad$ char. eqn. $r^2 - \alpha^2 = 0 \qquad r = \pm\alpha$

$\phi(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$, easier

$\phi(x) = d_1 \cosh(\alpha x) + d_2 \sinh(\alpha x) \qquad \left(c_1 = \frac{d_1 + d_2}{2}, \; c_2 = \frac{d_1 - d_2}{2}\right)$

$\phi(0) = d_1 = 0 \qquad \phi(L) = d_2 \sinh(\alpha L) = 0 \Rightarrow d_2 = 0$

only trivial soln.

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<u>case 1</u> $\lambda > 0$, let $\lambda = \alpha^2$

$$\phi'' + \alpha^2 \phi = 0 \qquad \text{char. eqn} \quad r^2 + \alpha^2 = 0 \qquad r = \pm i\alpha$$

$$\therefore \ \phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

$$\phi(0) = c_1 = 0$$

$$\phi(L) = c_2 \sin(\alpha L) = 0$$

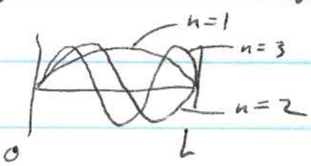
For nontrivial, want $\sin(\alpha L) = 0$

Obtain eigenvalues $\alpha = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$

or $\lambda_n = \frac{n^2\pi^2}{L^2}$, $n = 1, 2, 3$

Eigenfcns $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ nontrivial fcn satisfying ODE & B.C.'s

<u>Note</u>: $\phi_n(x)$ has $n-1$ zeroes in $0 < x < L$, which later we prove is a general property



<u>Product Soln. & Principle of Superposition</u>

Our product solns become

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2} \qquad n = 1, 2, \ldots$$

<u>Example</u> $u_t = k u_{xx}$ $t > 0$, $0 < x < 1$

B.C. $u(0,t) = 0$ $\quad u(1,t) = 0$ $\quad t > 0$

I.C. $u(x,0) = 3\sin(2\pi x) + 5\sin(5\pi x)$

<u>Soln</u> Sep. Var $u(x,t) = \phi(x) G(t)$

$$\phi G' = k \phi'' G \implies \frac{G'}{kG} = \frac{\phi''}{\phi} = -\lambda \quad , \ \phi(0) = 0 = \phi(1)$$

$$\phi'' + \alpha^2 \phi = 0 \qquad \phi(x) = b_n \sin(n\pi x) \qquad \alpha = n\pi, \ n = 1, 2, \ldots$$

$$u_n(x,t) = b_n \sin(n\pi x) e^{-kn^2\pi^2 t}$$

$$u(x,0) = 3\sin(2\pi x) + 5\sin(5\pi x) \implies b_2 = 3, \ b_5 = 5 \qquad b_n = 0 \ \ n \neq 3,5$$

$$u(x,t) = 3\sin(2\pi x) e^{-4k\pi^2 t} + 5\sin(5\pi x) e^{-25k\pi^2 t}$$

<u>Superposition (Extended)</u> The principle of superposition can be extended to show that if $u_1, u_2, \dots, u_M$ are solns of a linear homogeneous problem, then any linear combination

$$c_1 u_1 + c_2 u_2 + \dots + c_M u_M$$

is also a solution.

It follows that for the homogeneous heat problem

$$u_t = k\, u_{xx}, \quad u(0,t) = 0 \ \& \ u(L,t) = 0$$

$$u(x,t) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{k n^2 \pi^2 t}{L^2}}$$

satisfying the I.C.

$$u(x,0) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

I.C.

This works for any I.C. that is the finite sum of appropriate sine functions

What if $f(x)$ is <u>not</u> a finite linear combination of approp. sine fcns.?

<u>Chap 3 Fourier series</u>

1. Any function $f(x)$ with reasonable restrictions can be approximated by a linear combination of $\sin\left(\frac{n\pi x}{L}\right)$
2. Approx improves with M increasing
3. If we consider the limit as $M \to \infty$, then with some restrictions the eigenfcns $\sin\left(\frac{n\pi x}{L}\right)$ in the right combination converges to $f(x)$.

<u>Fourier Series</u> (Chap 3 for details) Many theoretical issues

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$\Rightarrow$ soln to heat problem is $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{k n^2 \pi^2 t}{L^2}}$

Orthogonality of Sines

How do we find B_n?

Assume $$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Assume std. math operations (linearity) hold for infinite series

Consider $$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & m\neq n \\ L/2 & m=n \end{cases}$$ (Use Maple)

(back 15)

This shows $\sin\frac{n\pi x}{L}$ is orthogonal to $\sin\frac{m\pi x}{L}$

<u>Orthogonality</u> Whenever $\int_0^L A(x)B(x)\,dx = 0$ (fcn inner prod) we say that the fcns $A(x)$ and $B(x)$ are <u>orthogonal</u> over the interval $[0,L]$.

The set of eigenfcns for the BVP $\phi'' + \lambda\phi = 0$, $\phi(0)=0$, $\phi(L)=0$ are orthogonal. We'll generalize in Chap. 5 to any Sturm-Liouville problem, which arise naturally in PDE's.

From above

$$f(x)\sin\left(\frac{m\pi x}{L}\right) = \sin\left(\frac{m\pi x}{L}\right)\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$
$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)$$

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

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$$= B_m\left(L/2\right)$$

Thus, $$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$ Fourier coef.

(Note: Average over full period of sine or cosine squared is $\frac{1}{2}$.)

Example $u_t = k\,u_{xx}$, $t > 0$, $0 < x < L$

B.C. $u(0,t) = 0$, $u(L,t) = 0$

I.C. $u(x,0) = 100$ $0 < x < L$

Soln From before

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{k n^2\pi^2 t}{L^2}}$$

Fourier coef.

$$B_n = \frac{2}{L}\int_0^L 100 \sin\frac{n\pi x}{L}\,dx = \frac{200}{L}\left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Big|_0^L$$

$$= \frac{200}{n\pi}\Big(1 - \underbrace{\cos(n\pi)}_{(-1)^n}\Big) = \begin{cases} 0 & n \text{ even} \\ \frac{400}{n\pi} & n \text{ odd} \end{cases}$$

Physical interpretation Note the decaying exponential gets large faster for $n > 1$

$\therefore$ $u(x,t) \simeq \frac{400}{\pi}\sin\left(\frac{\pi x}{L}\right)e^{-\frac{k\pi^2 t}{L^2}}$ $\Leftarrow$ dominant mode

from first eigenfcn.

1/29 $\rightarrow$ Show Matlab heatjn

Maple heat1d

```
% Generating solutions to the heat flow equation
% in an infinite slab of thickness W

format compact;
W = 10;                         % width of plate
Temp = 200;                     % temperature of RHS of slab
tfin = 20;                      % final time
alpha = 1;                      % heat coef of the medium

NptsX=151;
NptsT=151;
Nplots=500; Nrows=5;                    % number of columns
Nplots=540; Nrows=6;                    % number of columns
plotsols=0;
Ncols=Nplots/Nrows;
Nskip=2;                        % use every Nskip coefs
x=linspace(0,W,NptsX);
t=linspace(0,tfin,NptsT);
[X,T]=meshgrid(x,t);


fs=8;
figure(3)
clf

U0t = Temp/W*x;                 % initial theoretical steady state distribution
b=zeros(1,Nplots);

% Fourier coefs for U0(x) temperature
for jj=1:Nskip:Nplots
  b(jj) = (2*Temp)/(jj*pi)*(-1)^(jj-1);                  % Triangular IC
%  b(jj) = -200*((-1)^(jj)+1)/(jj*pi);
  b(jj) = Temp*(1-cos(jj*pi))/(jj*pi);                   % Square IC
end

b(1:Nskip:Nplots)

U=zeros(NptsT,NptsX);                   % initial steady state distribution (Fourier)

for jj=1:Nskip:Nplots           % build Fourier series
  Un=b(jj)*exp(-(jj*pi*alpha/W)^2*T).*sin(jj*pi*X/W);
  U=U+Un;
  if(plotsols==1)
   subplot(2*Nrows,Ncols,jj+Ncols*floor((jj-1)/Ncols))
   set(gca,'FontSize',[fs]);
   surf(X,T,Un);
   shading interp
   %colormap(gray)
   zl = ['B_{',num2str(jj),'} \phi_{',num2str(jj),'}(x)'];
   xlabel('x','Fontsize',fs); ylabel('t','Fontsize',fs); zlabel(zl,'Fontsize',fs);axis tight
   view([12 46])
   if(jj==1)
     ax=axis;
     ax(5)=-ax(6)/3;
   end
   axis(ax);
   set(gca,'FontSize',[fs]);

   subplot(2*Nrows,Ncols,jj+Ncols*floor((jj-1)/Ncols)+Ncols)
   set(gca,'FontSize',[fs]);
   surf(X,T,Un);
   shading interp
   colormap(jet)
   view([0 90])
   xlabel('x','Fontsize',fs); ylabel('t','Fontsize',fs); zlabel('z','Fontsize',fs);axis tigh
   set(gca,'FontSize',[fs]);
   %fprintf('Press any key to continue...\n');
   %pause
   pause(0.0001)
  end
```

2.4 Other Examples Heat Eqn.

2/2

2.4.1 Heat in Rod w/ insulated B.C.'s

$u_t = k\, u_{xx} \qquad t > 0 \qquad 0 < x < L$

B.C. $u_x(0,t) = 0 \qquad u_x(L,t) = 0$

I.C. $u(x,0) = f(x)$

Sep. Var $\quad u(x,t) = \phi(x)\, G(t)$

$$\phi G' = k \phi'' G \implies \frac{G'}{kG} = \frac{\phi''}{\phi} = -\lambda$$

$$G(t) = c\, e^{-\lambda k t}$$

BVP $\quad \phi'' + \lambda \phi = 0 \qquad \phi'(0) = 0 \qquad \phi'(L) = 0$

i) If $\lambda = -\alpha^2 < 0 \qquad \phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$

$\phi'(x) = \alpha\,(c_1 \sinh(\alpha x) + c_2 \cosh(\alpha x)) \qquad \phi'(0) = \alpha c_2 = 0 \Rightarrow c_2 = 0$

$\phi'(L) = \alpha c_1 \sinh(\alpha L) = 0 \Rightarrow c_1 = 0 \qquad$ trivial soln.

ii) If $\lambda = 0 \qquad \phi(x) = c_1 x + c_2 \qquad \phi'(0) = c_1 = \phi'(0) = 0 \quad \therefore c_1 = 0$

$\phi_0(x) = c_2$ is e.f. for e.v. $\lambda_0 = 0$

iii) If $\lambda = \alpha^2 > 0 \qquad \phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$

$\phi'(x) = \alpha\,(-c_1 \sin(\alpha x) + c_2 \cos(\alpha x)) \qquad \phi'(0) = 0 = \alpha c_2 \Rightarrow c_2 = 0$

$\phi'(L) = -c_1 \alpha \sin(\alpha L) = 0 \implies \alpha = \frac{n\pi}{L}$

$\therefore$ e.v.'s $\lambda = \frac{n^2\pi^2}{L^2}, \; n = 1, 2, \ldots \qquad$ e.f. $\phi_n(x) = \cos\frac{n\pi x}{L}$

Prod. Soln. (Superposition)

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-\frac{k n^2 \pi^2 t}{L^2}}$$

Orthogonality $\quad \int_0^L \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \neq 0 \\ L & n = m = 0 \end{cases} \qquad$ Maple?

<u>Fourier coef</u> $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

<u>Show?</u>

<u>Steady-State</u>

$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L}\int_0^L f(x)\,dx$ = ave. temp dist.

<u>Heat Conduction in Thin Circular Ring</u>

Insulated Thin Wire

$u_t = k\,u_{xx}$, $t>0$, $-L<x<L$

<u>B.C.</u> $u(-L,t) = u(L,t)$ $u_x(-L,t) = u_x(L,t)$, $t>0$ <u>Periodic B.C.'s</u>

<u>I.C.</u> $u(x,0) = f(x)$ $-L<x<L$

<u>S-L Prob</u> $\phi'' + \lambda\phi = 0$, $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$

If $\lambda = 0$, $\phi(x) = c_1 x + c_2$ B.C. $\Rightarrow$ e.f. $\phi(x) = 1$

Easily show $\lambda < 0 \Rightarrow$ trivial soln.

If $\lambda = \alpha^2 > 0$, $\phi(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$

$\phi(-L) = c_1\cos(-\alpha L) + c_2\sin(-\alpha L) = c_1\cos(\alpha L) - c_2\sin(\alpha L)$

$\phi(L) = c_1\cos(\alpha L) + c_2\sin(\alpha L)$

$\therefore\ \phi(-L) = \phi(L) \Rightarrow c_2\sin(\alpha L) = 0$ $\therefore\ \alpha_n = \frac{n\pi}{L}$

$\phi'(-L) = -c_1\alpha\sin(-\alpha L) + c_2\alpha\cos(\cancel{-}\alpha L) = c_1\alpha\sin(\alpha L) + c_2\alpha\cos(\alpha L)$

$\phi'(L) = -c_1\alpha\sin(\alpha L) + c_2\alpha\cos(\alpha L)$

$\therefore\ \phi'(-L) = \phi(L) \Rightarrow c_1\alpha\sin(\alpha L) = 0$ or $\alpha_n = \frac{n\pi}{L}$

Both c_1 + c_2 arb. e.f. $\phi_n(x) = \cos\frac{n\pi x}{L}$ + $\sin\frac{n\pi x}{L}$, $n = 1, 2, \ldots$

Thus,

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2 kt} + b_n\sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n\pi}{L}\right)^2 kt}\right)$$

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with

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

2.4.1
2.4.2

Orthogonality

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\left(\frac{m\pi x}{L}\right)dx = 0 \qquad \forall\, n > 0,\ m \geq 0$$

Fourier coef

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Summary p. 69

2.5 Laplace's Eqn.

Steady-State in 2D Rectangle

$$u_{xx} + u_{yy} = 0 \qquad 0 < x < L, \quad 0 < y < H$$

B.C. $u(0,y) = g_1(y)$, $u(L,y) = g_2(y)$
$u(x,0) = f_1(x)$, $u(\cancel{x}, H) = f_2(x)$ } Dirichlet B.C.

Eqn. is Linear homogeneous, B.C. Linear, not homogeneous

Break into 4 prob., each 1 nonhomogeneous B.C.

$u = u_1 + u_2 + u_3 + u_4$

Consider $u_{xx} + u_{yy} = 0 \quad 0 < x < L, \; 0 < y < H$

$u(0,y) = g_1(y)$, $u(L,y) = 0$, $u(x,0) = 0$, $u(x,H) = 0$

Use Sep of Var.

$$u(x,y) = h(x)\,\phi(y)$$

B.C. $\Rightarrow h(L) = 0,\; \phi(0) = 0,\; \phi(H) = 0$

PDE $\Rightarrow h''\phi + h\phi'' = 0 \Rightarrow \dfrac{h''}{h} = -\dfrac{\phi''}{\phi} = \lambda$

SL Prob $\quad \phi'' + \lambda\phi = 0, \quad \phi(0) = 0 \quad \phi(H) = 0$

From before, obtain $\lambda = \alpha^2 > 0$

e.v.'s $\lambda_n = \dfrac{n^2\pi^2}{H^2}$, e.f.'s $\phi_n(y) = \sin\left(\dfrac{n\pi y}{H}\right)$, $n = 1, 2, \ldots$

x-eqn $\quad h'' - \dfrac{n^2\pi^2}{H^2} h = 0 \Rightarrow h(x) = c_1 \cosh\left(\dfrac{n\pi x}{H}\right) + c_2 \sinh\left(\dfrac{n\pi x}{H}\right)$

Better written $h(x) = a_1 \cosh\left(\frac{n\pi}{H}(x-L)\right) + a_2 \sinh\left(\frac{n\pi}{H}(x-L)\right)$

Since $h(L) = 0 \Rightarrow a_1 = 0$

$\therefore \; h_n(x) = a_2 \sinh\left(\frac{n\pi}{H}(x-L)\right)$

Thus, $\quad u_n(x,y) = A \sinh\left(\frac{n\pi}{H}(x-L)\right) \sin\left(\frac{n\pi y}{H}\right) \quad n = 1, 2, \ldots$

so by extended superposition

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{H}(x-L)\right) \sin\left(\frac{n\pi y}{H}\right)$$

From nonhomo B.C.

$$g_1(y) = u(0,y) = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{n\pi L}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

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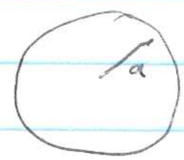
Fourier coef

$$A_n = \frac{-2}{H \sinh\left(\frac{n\pi L}{H}\right)} \int_0^H g_1(y) \sin\left(\frac{n\pi y}{H}\right) dy$$

Similar for other ~~B~~ B.C.'s, then add solutions.

Laplace Eqn. - Circular Disk



$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

2 B.C in r — B.C. $u(a, \theta) = f(\theta)$

Implicit $|u(0,\theta)| < \infty$ bdd at origin (singularity)

2 B.C in θ — Periodic B.C. $u(r,-\pi) = u(r,\pi)$, $u_\theta(r,-\pi) = u_\theta(r,\pi)$ } homogeneous

Sep. Var $u(r,\theta) = \phi(\theta) G(r)$

$$\phi(-\pi) = \phi(\pi), \quad \phi'(-\pi) = \phi'(\pi)$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi + \frac{1}{r^2} G(r)\phi'' = 0$$

$$\Rightarrow \frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{\phi''}{\phi} = \lambda$$

SL Prob $\phi'' + \lambda\phi = 0$, $\phi(-\pi) = \phi(\pi)$, $\phi'(-\pi) = \phi'(\pi)$

Recall circular wire.

If $\lambda < 0$ only trivial soln.

If $\lambda = 0$, $\phi(\theta) = c_1\theta + c_2$ B.C. $\Rightarrow c_1 = 0$ $\therefore$ $\phi_0(\theta) = 1$ e.f. for e.v. $\lambda = 0$

If $\lambda = \alpha^2 > 0$

$$\phi(\theta) = c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta)$$

$\phi(-\pi) = c_1 \cos(-\alpha\pi) + c_2 \sin(-\alpha\pi)$

$\quad = c_1 \cos(\alpha\pi) - c_2 \sin(\alpha\pi)$

$\phi(\pi) = c_1 \cos(\alpha\pi) + c_2 \sin(\alpha\pi) \quad \Rightarrow \quad \sin(\alpha\pi) = 0$

$\therefore \alpha = n \qquad$ e.f. $\sin(n\theta)$

$\phi'(-\pi) = -c_1 \alpha \sin(-\alpha\pi) + c_2 \alpha \cos(-\alpha\pi)$

$\quad = c_1 \alpha \sin(\alpha\pi) + c_2 \alpha \cos(\alpha\pi)$

$\phi'(\pi) = -c_1 \alpha \sin(\alpha\pi) + c_2 \alpha \cos(\alpha\pi) \quad \Rightarrow \quad \sin(\alpha\pi) = 0$

$\therefore \alpha = n \qquad$ e.f. $\cos(n\theta)$

Thus, e.v. $\lambda_n = n^2$, e.f. $\phi_n(\theta) = c_1 \cos(n\theta) + c_2 \sin(n\theta)$

<u>r - Eqn</u> $\quad \frac{r}{G} \frac{d}{dr}\left(r \frac{dG}{dr}\right) = n^2$

$\Leftrightarrow \quad r^2 G'' + r G' - n^2 G = 0$

<u>Euler's Eqn</u> Let $G(r) = c r^\alpha \qquad G' = c\alpha r^{\alpha-1} \qquad G'' = c\alpha(\alpha-1) r^{\alpha-2}$

$c\alpha(\alpha-1) r^\alpha + c\alpha r^\alpha - n^2 c r^\alpha = c r^\alpha(\alpha^2 - n^2) = 0$

$\Rightarrow \alpha = \pm n, \quad n \neq 0 \qquad$ if $n = 0 \quad G(r) = c_1 + c_2 \ln(r)$

$\therefore G(r) = c_1 r^n + c_2 r^{-n} \qquad$ integrate $\frac{d}{dr}\left(r \frac{dG}{dr}\right) = 0$

Since $G(r)$ bdd at $r = 0$, $c_2 = 0$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} B_n r^n \sin(n\theta)$$

$0 \le r < a$

$-\pi < \theta \le \pi$

Satisfy nonhomo B.C.

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$$f(\theta) = u(a,\theta) = A_0 + \sum_{n=1}^{\infty} \left(A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta)\right)$$

$-\pi < \theta \le \pi$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta$$

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta \qquad B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta$$

Chap 3 Fourier Series

Introduction

Separation of Variable worked provided we could write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Questions:

1. Does the infinite series converge?
2. Does it converge to $f(x)$?
3. Is the resulting infinite series really a soln. of the PDE (and its subsidiary conditions)?

Mathematically, these are all difficult problems, yet these solns have worked well since the early 1800s.

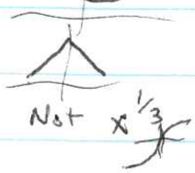
Begin by restricting class of $f(x)$ that we'll consider

A fcn $f(x)$ is piecewise smooth (on some interval) iff $f(x)$ is continuous and $f'(x)$ is continuous on a finite collection of sections of the given interval.

The only discontinuities allowed are jump discontinuities.

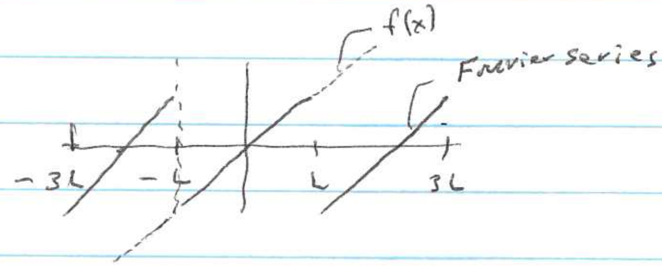
A fcn $f(x)$ has a jump discontinuity at a pt $x = x_0$ if the limit from the right $[f(x_0^+)]$ and the limit from the left $[f(x_0^-)]$ both exist and are not equal.

Piecewise smooth allows only a finite number of jump discontinuities in the fcn, $f(x)$, and its derivative, $\frac{df}{dx}$



The <u>Fourier series</u> of $f(x)$ on interval $-L \le x \le L$ is periodic with <u>period $2L$</u>. The function $f(x)$ doesn't need to be periodic.

e.g. $f(x) = x$



Above shows we must distinguish between $f(x)$ & its Fourier series.

Convergence T

Fourier series for $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

This infinite series may not converge & if it converges, it may not converge to $f(x)$. The Fourier coefficients are defined (using orthogonality)

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

The Fourier series of $f(x)$ over $-L \le x \le L$ is defined to be the infinite series above with the Fourier coefficients above.

Clearly, the coefficients must be defined, e.g. $\left|\int_{-L}^{L} f(x)\,dx\right| < \infty$ for a_0 to exist. (No Fourier series for $f(x) = 1/x^2$.)

When $\int_{-L}^{L} f(x)\,dx$ exists, the infinite series may fail to converge.

When it converges, it may not converge to $f(x)$.

We write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$\uparrow$ has the Fourier series

Fourier's Th

<u>Convergence Thm for Fourier series</u> If $f(x)$ is piecewise smooth on the interval $-L \le x \le L$, then the Fourier series of $f(x)$ converges to the

1. periodic extension of $f(x)$, where the periodic extension is continuous
2. average of the two limits, usually

$$\frac{1}{2}\left[f(x+) + f(x^-)\right]$$

where the periodic extension has a jump discontinuity.

<u>Proof available through references in book.</u>

Thus, Fourier series converges to pts of cont. $-L < x < L$, to midpts of a jump discont., & average of values at periodic extensions.

<u>e.g.</u> $f(x) = \begin{cases} 0 & x < 1 \\ 1 & x > 1 \end{cases}$

Fourier series with $L = 2$ (arb. L)

$a_0 = \frac{1}{4}$

$$a_n = \frac{\overset{0}{\sin(n\pi)} - \sin\left(\frac{n\pi}{2}\right)}{n\pi} \qquad b_n = \frac{\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)}{n\pi}$$

2/7 See computer simulation.

Fourier Sine Series

If $f(x)$ is an odd fcn., then $a_0 = a_n = 0$ & only sine series remains

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Saw this for solutions of heat eqn. $0 < x < L$ with $u(0,t) = u(L,t) = 0$

Sine series produces an odd extension of $f(x)$

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \quad 0 \le x \le L \qquad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

If $f(x)$ is an even fcn, then $b_n = 0$ & only cosine series remains

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 \le x \le L$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

This is forced by insulated B.C. as seen before

Gibbs phenomenon

Consider odd extension $f(x) = 100$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) = \frac{200}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & n \text{ even} \\ \frac{400}{n\pi} & n \text{ odd} \end{cases}$$

What about convergence?

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

At $x = 0$, series $= 0$ as each term is 0 (similar at $x = L$)

Convergence Th claims $\rightarrow 100$ for each $0 < x < L$

Alternating series

Consider $x = L/2$,

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{2}\right) = \frac{400}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

Euler's formula gives $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$
(inefficient way to compute π)

Harder to show convergence for other values of $x \in (0, L)$
Convergence easily visualized as worst near jump discont.
For any finite sum in the series, near $x = 0$, the soln starts at 0, then shoots up beyond 100, the primary overshoot.

This overshoot is an example of the Gibbs phenomenon.
For large n, in general, there is an overshoot of $\sim 9\%$ of the jump discontinuity.
Only occurs ~~for~~ a finite series at a jump discontinuity.

Cont. Fourier Series

For a piecewise smooth $f(x)$, the Fourier series of $f(x)$ is cont. and converges to $f(x)$ for $x \in [-L, L]$
iff $f(x)$ is cont. and $f(-L) = f(L)$.

For piecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is cont. & converges to $f(x)$ for $x \in [0, L]$
iff $f(x)$ is cont.

For piecewise smooth $f(x)$, the Fourier sine series of $f(x)$ is cont and converges to $f(x)$ for $x \in [0, L]$
iff $f(x)$ is cont. and both $f(0) = 0$ and $f(L) = 0$.

3.4 Term-by-term Differentiation of Fourier Series

We solved $u_t = k\,u_{xx}$, $u(0,t)=0$, $u(L,t)=0$, $u(x,0)=f(x)$

obtaining

$$u(x,t)=\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n^2\pi^2}{L^2}\right)kt}$$

if we can extend finite series to infinite series.

If $f(x)$ is piecewise smooth, earlier Th shows this Fourier series converges to the I.C.

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Suppose we can differentiate term-by-term

$$\frac{\partial u}{\partial t} = -\sum_{n=1}^{\infty} k\left(\frac{n^2\pi^2}{L^2}\right)B_n \sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n^2\pi^2}{L^2}\right)kt}$$

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty}\left(\frac{n^2\pi^2}{L^2}\right)B_n \sin\left(\frac{n\pi x}{L}\right)e^{-\left(\frac{n^2\pi^2}{L^2}\right)kt}$$

so satisfies heat eqn.

<u>Problem</u> <u>Counterexample</u>

$$x = 2\sum_{n=1}^{\infty}\frac{L}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right) \quad \text{on } 0 \le x < L$$

Differentiating term-by-term gives

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\cos\left(\frac{n\pi x}{L}\right)$$

but this is not the cosine series for $f(x)=1$ (derivative of x)

(This series fails to converge anywhere since the n^{th} term doesn't approach zero.)

<u>When is term-by-term differentiation justified?</u>

<u>Th</u> A Fourier series that is cont. can be differentiated term-by-term if $f'(x)$ is piecewise smooth.

Cor If $f(x)$ is piecewise smooth, then the Fourier series of a cont. fcn $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

The result of term-by-term differentiation is the Fourier series of $f'(x)$, which may not be cont.

Fourier cosine series

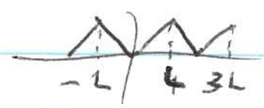
Th If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a fcn $f(x)$ can be differentiated term by term.

$$\therefore \quad f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \qquad 0 \le x \le L$$

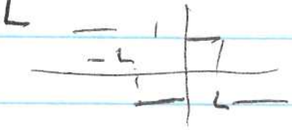
(equality means converges $\forall\ x \in [0, L]$.

$$\Rightarrow \quad f'(x) \sim -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) A_n \sin\left(\frac{n\pi x}{L}\right)$$

e.g. $x = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos\left(\frac{n\pi x}{L}\right) \qquad 0 \le x \le L$



$1 \sim \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) \qquad 0 < x < L$



Giving the correct derivative.

Fourier sine series

Th If $f'(x)$ is piecewise smooth, then the Fourier sine series of a fcn $f(x)$ can only be differentiated term by term if $f(0) = 0$ and $f(L) = 0$.

Pf Available p120-121

Generalizes

Th If $f'(x)$ is piecewise smooth, then the Fourier sine series of a cont. fcn $f(x)$

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

cannot, in general be differentiated term by term. However,

$$f'(x) \sim \frac{1}{L}\left[f(L)-f(0)\right] + \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}B_n + \frac{2}{L}\left((-1)^n f(L) - f(0)\right)\right] \cos\frac{n\pi x}{L}$$

Ex Reconsider $f(x) = x$

$$x \sim 2\sum_{n=1}^{\infty} \frac{L}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{L}$$

Since $f(0) = 0$, $f(L) = L$ $\Rightarrow$ $\frac{1}{L}\left(f(L)-f(0)\right] = 1$ $\Rightarrow A_0 = 1$

Also, $\frac{n\pi}{L}B_n = 2(-1)^{n+1}$, but $2(-1)^{n+1} + 2(-1)^n = 0$ $\Rightarrow A_n = 0$

Thus, obtain correct expression for $f'(x)$.

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Method of e.f. expansion

Assume heat eqn. w/ homo. b.c.'s (Dirichlet)

Claim

$$u(x,t) \sim \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{L}$$

where I.C.'s give

$$f(x) \sim \sum_{n=1}^{\infty} B_n(0) \sin\frac{n\pi x}{L} \quad \text{w/} \quad B_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Can we differentiate term by term to satisfy heat eqn $u_t = k u_{xx}$

Need to compute 2 derivs w.r.t. x

If $u(x,t)$ is cont., then Fourier sine series can be diff. term by term provided $u(0,t)=0$ and $u(L,t)=0$ (homo. b.c.)

$$\frac{\partial u}{\partial x} \sim \sum_{n=1}^{\infty} \frac{n\pi}{L} B_n(t) \cos\left(\frac{n\pi x}{L}\right)$$

This is a cosine series, so provided $\frac{\partial u}{\partial x}$ is cont, it canbe diff term by term

$$\frac{\partial^2 u}{\partial x^2} \sim -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 B_n(t) \sin\left(\frac{n\pi x}{L}\right)$$

Thus, needed homo. b.c., but otherwise satisfied term-by-term diff.

Need $\partial u/\partial t \sim \sum_{n=1}^{\infty} \frac{dB_n}{dt} \sin \frac{n\pi x}{L}$.

If term by term is justified

$$\frac{dB_n}{dt} = -k\left(\frac{n\pi}{L}\right)^2 B_n(t)$$

so $$B_n(t) = B_n(0)\, e^{-\frac{n^2\pi^2}{L^2} k t}$$

<u>Th</u> The Fourier series of a cont. fcn $u(x,t)$

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left(a_n(t) \cos \frac{n\pi x}{L} + b_n(t) \sin \frac{n\pi x}{L}\right)$$

can be diff. term by term w.r.t. t

$$\frac{\partial u(x,t)}{\partial t} = a_0'(t) + \sum_{n=1}^{\infty}\left(a_n'(t) \cos\left(\frac{n\pi x}{L}\right) + b_n'(t) \sin\left(\frac{n\pi x}{L}\right)\right)$$

if $\frac{\partial u}{\partial t}$ is piecewise smooth.

This justifies use of separation of variables + our soln.

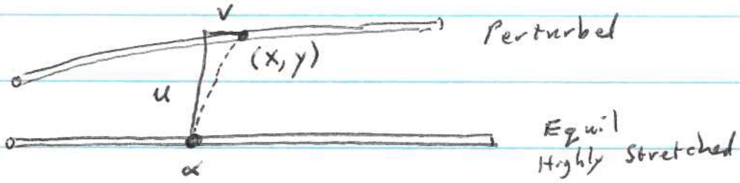
3.5 Term by Term Integration

<u>Th</u> A Fourier series of piecewise smooth $f(x)$ can always be integrated term by term and the result is a convergent infinite series that always converges to the integral of $f(x)$ for $-L \le x \le L$ (even if the original Fourier series has jump discontinuities).

3.6 Complex (Return to Later)

Wave Eqn: Vibrating Strings & Membranes

4.2 Derivation

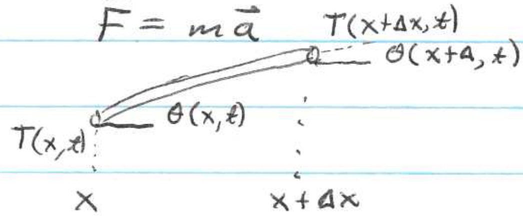


Examine particle at position α in highly stretched string
Assume small displacement ⇒ only vert. motion

$$y = u(x,t) \qquad \text{vertical displacement}$$

Newton's Law. Consider an infinitesimally small thin segment of string between x & $x + \Delta x$.
Assume mass density $\rho_0(x)$ ∴ mass of segment $\rho_0(x)\Delta x$

$$\vec{F} = m\vec{a}$$



Consider forces acting on string (e.g. gravity, end forces) "body" forces
Assume perfectly flexible ⇒ no bending resistance
⇒ force tangent to string at all pts.
Tension is the tangential force

$$\frac{dy}{dx} = \frac{\partial u}{\partial x} = \tan(\theta(x,t))$$

Newton's Law

Vertical Body ~~forces~~ accelerat

$$\underbrace{\rho_0(x)\Delta x}_{m}\underbrace{\frac{\partial^2 u}{\partial t^2}}_{a} = T(x+\Delta x, t)\sin(\theta(x+\Delta x, t)) - T(x,t)\sin(\theta(x,t)) + \rho_0(x)\Delta x\, Q(x,t)$$

$$\lim_{\Delta x \to 0} \Rightarrow \quad \rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x,t)\sin(\theta(x,t))\right] + \rho_0(x)\,Q(x,t)$$

For θ small $\quad \frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta$

String Eqn.

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial u}{\partial x}\right) + \rho_0(x)\,Q(x,t)$$

Perfectly elastic string $T(x,t) \approx T_0$ constant (stretching almost uniform along string.

$$\rho_0(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\rho_0(x)$$

If body force small + density constant

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T_0}{\rho_0}$$

4.3 Boundary Conditions

Skip section - Odd conditions given to show Robin-type B.C.

4.4 Vibrating String w/ Fixed Ends

$$u_{xx} = c^2 u_{tt}, \quad t > 0 \quad 0 < x < L \qquad c^2 = T_0/\rho_0$$

B.C. $u(0,t) = 0 \qquad u(L,t) = 0$

I.C. $u(x,0) = f(x) \qquad u_t(x,0) = g(x)$ position + velocity

Sep. of Var $u(x,t) = \phi(x)\,h(t)$

$$\phi'' h = c^2 \phi h'' \Rightarrow \frac{1}{c^2 h}h'' = \frac{\phi''}{\phi} = -\lambda$$

B.C. $\Rightarrow \phi(0) = 0 \qquad \phi(L) = 0$

SL Prob $\phi'' + \lambda\phi = 0, \quad \phi(0) = \phi(L) = 0$

As before $\lambda \le 0 \Rightarrow$ trivial soln

Take $\lambda = \alpha^2 > 0$

$$\phi(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x) \qquad \phi(0) = 0 \Rightarrow c_1 = 0$$

$$\phi(L) = 0 \Rightarrow \alpha = \frac{n\pi}{L}$$

e.v. $\lambda_n = \frac{n^2\pi^2}{L^2}$ e.f. $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

Other Problem $\quad h'' = -\frac{n^2\pi^2}{L^2}c^2 h$

$$\therefore \; h_n(t) = c_1 \cos\left(\frac{n\pi c t}{L}\right) + c_2 \sin\left(\frac{n\pi c t}{L}\right)$$

Superposition

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)\right)$$

Initial Position

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Initial Velocity $\qquad$ term by term diff

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-A_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)\sin\frac{n\pi c t}{L} + B_n\frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)\right)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Write

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi c t}{L}\right) + B_n \sin\left(\frac{n\pi c t}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

Interpret in context of musical instruments

Each value n gives a normal mode of vibration

Intensity depends on amplitude

$$A_n\cos(\omega t) + B_n \sin(\omega t) = \sqrt{A_n^2 + B_n^2}\,\sin(\omega t + \theta) \qquad \theta = \arctan\left(A_n/B_n\right)$$

$$A\sin(\omega t + \theta) = A\left(\cos(\omega t)\sin(\theta) + \sin(\omega t)\cos(\theta)\right)$$

$$A_n = A\sin\theta, \quad B_n = A\cos\theta \implies A^2 = A_n^2 + B_n^2 \qquad \tan\theta = A_n/B_n$$

Time dependence is <u>simple harmonic</u> with <u>circular frequency</u> $= \frac{n\pi c}{L}$
(number of oscillations in 2π units of time) $c = \sqrt{T_0/\rho_0}$
Sound produced consists of superposition of the infinite number of <u>natural frequencies</u> ($n = 1, 2, \ldots$

The normal mode $n=1$ is called the 1st harmonic or fundamental
This mode has <u>circular frequency</u> of $\pi c/L$
The higher natural freq produces higher pitch
Fundamental frequency varied by changing $c = \sqrt{T_0/\rho_0}$
Tune instrument by changing Tension T_0
Different ρ_0 for different strings (range of notes)
Musician varies pitch by varying length L (clamping string)
Higher harmonics for stringed instruments are all integral multiples (pleasing to ear)

At each t, each mode looks like a simple oscillation in x
The amplitude varies in time — <u>Standing Wave</u>
For a standing wave,

$$\sin\frac{n\pi x}{L}\sin\frac{n\pi c t}{L} = \underbrace{\frac{1}{2}\cos\frac{n\pi}{L}(x-ct)}_{\substack{\text{wave traveling to right}\\ \text{velocity } c}} - \underbrace{\frac{1}{2}\cos\frac{n\pi}{L}(x+ct)}_{\substack{\text{wave traveling to left}\\ \text{velocity } -c}}$$

it decomposes into two travelling waves

By superposition principle

$$u(x,t) = R(x-ct) + S(x+ct)$$

examine in more detail later.

<u>Skip 4.5 & 4.6 Vibrating Membrane & Snell's Law</u>

Chap 5 Sturm-Liouville Eigenvalue Problems

5.2 Heat Flow in Nonuniform Rod

Spse we allow $c, \rho, + K_0$ to depend on x + let

$$Q = \alpha(x)\, u \qquad \text{prop. to temperature (Newton's cooling)}$$

Heat Eqn.

$$c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right) + \alpha u$$

Sep Var. $\quad u(x,t) = \phi(x)\, h(t)$

$$\frac{h'}{h} = \frac{1}{c\rho\phi} \frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \frac{\alpha}{c\rho} = -\lambda$$

Eqn. in x is

$$\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\phi = 0$$

This is a Sturm-Liouville BVP if homogeneous B.C.'s

Soln. to d.e. may be difficult to find.

Circularly Symmetric Heat Flow

$$\frac{\partial u}{\partial t} = k \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)$$

$$u(r,t) = \phi(r)\, h(t) \quad \Rightarrow \quad \frac{h'}{kh} = \frac{1}{r\phi} \frac{d}{dr}\left(r \frac{d\phi}{dr}\right) = -\lambda$$

S-L Prob $\quad \frac{d}{dr}\left(r \frac{d\phi}{dr}\right) + \lambda r \phi = 0 \quad$ + homo. B.C.'s

$u(a,t) = 0 \quad u(b,t) = 0 \quad$ annulus

circular gives singularity condition $|u(0,t)| < \infty$.

5.3 Sturm - Liouville E.V. Prob.

General S-L D.E.

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\,\phi + \lambda\,\sigma(x)\,\phi = 0$$

where λ is an eigenvalue & $a < x < b$

Examples to date

1. $p(x) = \sigma(x) = 1$, $q(x) = 0$ $\qquad \phi'' + \lambda\phi = 0$
2. Non uniform heat flow $\frac{d}{dx}\left(K_0\frac{d\phi}{dx}\right) + \alpha\phi + \lambda c\rho\,\phi = 0$

   $K_0 = p(x)$ $\quad \alpha = q(x)$ $\quad c\rho = \sigma(x)$
3. Circular symmetry $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0$

B.C.'s	Heat	String	
$\phi = 0$	Fixed zero temp.	fixed ends	Dirichlet (1st Kind)
$\phi' = 0$	Insulated	free	Neumann (2nd Kind)
$\phi' = \pm h\phi$	Newton cooling	elastic boundary	Robin (3rd Kind)
$\phi(-L) = \phi(L)$, $\phi'(L) = \phi'(L)$	Perfect Thermal contact		Periodic
$\lvert\phi(0)\rvert < \infty$	Bdd temp		Singularity

Regular SL E.V. Prob

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\,\phi + \lambda\sigma(x)\,\phi = 0 \qquad a < x < b$$

$\beta_1\,\phi(a) + \beta_2\,\phi'(a) = 0$ $\qquad$ exclude periodic & sing.

$\beta_3\,\phi(b) + \beta_4\,\phi'(b) = 0$

β_i real & $p(x)$, $q(x)$ & $\sigma(x)$ are real & cont. $a \le x \le b$ (including endpts) $p(x)$ & $\sigma(x) > 0$, $a \le x \le b$ (including endpts)

<u>Important Theorems</u> - state, prove some later

1. All e.v. λ are real.
2. $\exists$ infinite # e.v.'s $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$
   a. There is a smallest e.v. denoted λ_1
   b. There is not a largest e.v., i.e., $\lambda_n \to \infty$ as $n \to \infty$
3. Corresponding to each e.v. λ_n there is an e.f. $\phi_n(x)$
   $\phi_n(x)$ has exactly $n-1$ zeros for $a < x < b$
4. The e.f.'s $\phi_n(x)$ form a "complete" set, meaning that any piecewise smooth fcn $f(x)$ can be represented by a generalized Fourier series $f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x)$

   Furthermore, the infinite series converges to $[f(x+) + f(x-)]/2$ for $a < x < b$ (appropriate a_n).
5. E.f.'s corresponding to different e.v.'s are orthogonal relative to the weight fcn. $\sigma(x)$

$$\int_a^b \phi_n(x)\,\phi_m(x)\,\sigma(x)\,dx = 0 \quad \text{if} \quad \lambda_n \neq \lambda_m$$

6. Any eigenvalue can be related to its e.f. by the <u>Rayleigh quotient</u>

$$\lambda = \frac{-p\,\phi(x)\phi'(x)\Big|_a^b + \int_a^b \left[p\left(\frac{d\phi}{dx}\right)^2 - q\,\phi^2\right]dx}{\int_a^b \phi^2 \sigma\, dx}$$

where the B.C.'s may simplify this expression

Example $\phi'' + \lambda\phi = 0$, $\phi(0) = 0$, $\phi(L) = 0$

Found $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, \ldots$

1. Found real e.v.'s , need to establish not complex
2. $\lambda_1 = \left(\frac{\pi}{L}\right)^2$ and clearly $\lambda_n \to \infty$ as $n \to \infty$
3. Easily seen that $\phi_n(x)$ has $n-1$ zeros for $x \in (0, L)$
4. Established Fourier series for this SL problem
5. Showed orthogonality of ϕ_n
6. Reyleigh Quotient simplifies $\lambda = \dfrac{\int_0^L (\phi')^2\,dx}{\int_0^L (\phi)^2\,dx}$ shows > 0

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5.4 Nonuniform Rod

PDE: $c\rho \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0 \frac{\partial u}{\partial x}\right)$ c, ρ, K_0 non const possibly (fcn of x)

B.C. $u(0,t) = 0$ $\frac{\partial u}{\partial x}(L,t) = 0$

I.C. $u(x,0) = f(x)$

Sep. Var $u(x,t) = \phi(x)\,h(t)$

$$\frac{h'}{h} = \frac{\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right)}{c\rho\,\phi} = -\lambda$$

Time soln. $h(t) = c\,e^{-\lambda t}$

SL - Prob $\frac{d}{dx}\left(K_0 \frac{d\phi}{dx}\right) + \lambda c\rho\,\phi = 0$

B.C. $\phi(0) = 0$, $\phi'(L) = 0$

Thus $\Rightarrow$ infinite seq. e.v. λ_n & corresponding ϕ_n

Finding ϕ_n might be difficult, but solns exist.

Superposition principle gives

$$u(x,t) = \sum_{n=1}^{\infty} a_n \phi_n(x) e^{-\lambda_n t}$$

I.C. $\quad u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$

Thms $\Rightarrow$ $f(x)$ piecewise smooth

$$a_n = \frac{\int_0^L f(x)\,\phi_n(x)\,c(x)\,\rho(x)\,dx}{\int_0^L \phi_n^2(x)\,c(x)\,\rho(x)\,dx}$$

using orthogonality relation

$$\int_0^L \phi_n(x)\,\phi_m(x)\,c(x)\,\rho(x)\,dx = 0 \qquad n \neq m$$

For large time $\quad u(x,t) \approx a_1 \phi_1(x) e^{-\lambda_1 t} \quad$ shape of 1^{st} e.f.

By Rayleigh Quotient

$$\lambda = \frac{\int_0^L K_0(x)(\phi'(x))^2\,dx}{\int_0^L \phi^2(x)\,c(x)\,\rho(x)\,dx}$$

$\phi(0) = 0 \Rightarrow$ constant e.f. not allowable

$\therefore (\phi'(x))^2 > 0$, thus $\lambda > 0$

Since all e.v.'s > 0 soln. decays to zero

This is what we expect for physical problem (heat lost on left end).

5.5. Self-Adjoint Operators & S-L E.V. Problems

Regular SL-Prob

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right) + q(x)\phi + \lambda\sigma(x)\phi = 0$$

$$\beta_1 \phi(a) + \beta_2 \phi'(a) = 0$$
$$\beta_3 \phi(b) + \beta_4 \phi'(b) = 0$$

β_i real , p, q, σ real cont fcn. $a \le x \le b$ with $p(x) > 0$, $\sigma(x) > 0$

Won't prove 1. There are infinity many e.v.
2. Piecewise smooth fcn. can be expanded by e.f.
3. Each succeeding e.f. has an additional zero.

not reg. may have no e.v., but holds for most physical prob.

Linear Operator Let L be the linear differential operator

$$L = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$$

$$L(y) = \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y$$

SL - d.e. written $L(\phi) + \lambda\sigma(x)\phi = 0$

Lagrange's identity

$$u L(v) - v L(u) = u\left[\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] + q(x)v\right] - v\left[\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u\right]$$

$$\frac{d}{dx}\left(u\left(p\frac{dv}{dx}\right)\right) = u\frac{d}{dx}\left(p\frac{dv}{dx}\right) + \frac{du}{dx}\cdot p\frac{dv}{dx}$$
$$\frac{d}{dx}\left(v\left(p\frac{du}{dx}\right)\right) = v\frac{d}{dx}\left(p\frac{du}{dx}\right) + \frac{dv}{dx}\cdot p\frac{du}{dx}$$

subtract

$$= u\frac{d}{dx}\left[p(x)\left(\frac{dv}{dx}\right)\right] - v\frac{d}{dx}\left[p(x)\left(\frac{du}{dx}\right)\right]$$

$$= \frac{d}{dx}\left[p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right]$$

<u>Green's Formula</u> (integrating Lagrange's identity)

$$* \quad \int_a^b [uL(v) - vL(u)]\,dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$$

Need u, v, to be C^1 (continuously differentiable)

<u>Self-adjointness</u> S'pse u and v are any two fcn w/ the additional restriction that the boundary terms vanish

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0$$

<u>Thm</u> If u and v are any two fcns satisfying the same set of homogeneous b.c.'s of the regular Sturm-Liouville prob, then

$$\int_a^b [uL(v) - vL(u)]\,dx = 0.$$

The operator L satisfying this condition is <u>self-adjoint</u>.

S.S.1
Exercise to show for regular SL-Prob
Easy to show for periodic b.c.
Also easy if sing. at endpt. $p(0)=0$ with $\phi(0)$ bdd.

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<u>Orthogonality of e.f.</u>

Let λ_n & λ_m be distinct e.v. w/ corresponding e.f. ϕ_n & ϕ_m, so

$$L(\phi_n) + \lambda_n \sigma(x)\phi_n = 0$$
$$L(\phi_m) + \lambda_m \sigma(x)\phi_m = 0$$

$$\int_a^b [\phi_m L(\phi_n) - \phi_n L(\phi_m)]\,dx = (\lambda_m - \lambda_n)\int_a^b \phi_m \phi_n \sigma(x)\,dx$$

by Green's Formula

$$(\lambda_m - \lambda_n)\int_a^b \phi_m \phi_n \sigma\,dx = p(x)\left(\phi_m \frac{d\phi_n}{dx} - \phi_n\frac{d\phi_m}{dx}\right)\Big|_a^b$$

If ϕ_n + ϕ_m satisfy same homogeneous b.c.'s r.h.s $= 0$

$$\therefore \quad (\lambda_m - \lambda_n) \int_a^b \phi_m \phi_n \sigma \, dx = 0$$

Thus, e.f.'s corresponding to different e.v.'s are <u>orthogonal with weight, $\sigma(x)$.</u> (Don't forget $\sigma(x)$)

<u>Real e.v.</u> Spse λ complex e.v. and $\phi(x)$ corresponding e.f.

$$L(\phi) + \lambda \sigma \phi = 0$$

Take complex conjugate

$$\overline{L(\phi)} + \bar{\lambda} \sigma \bar{\phi} = 0$$

However $\overline{L(\phi)} = L(\bar{\phi})$, since coef. of L are real, so

$$L(\bar{\phi}) + \bar{\lambda} \sigma \bar{\phi} = 0$$

If λ is a complex e.v. w/ e.f. ϕ, then $\bar{\lambda}$ is also an e.v. w/ e.f. $\bar{\phi}$.

By orthogonality Th. above

$$(\lambda - \bar{\lambda}) \int_a^b \phi \bar{\phi} \sigma \, dx = 0$$

but $\phi \bar{\phi} = |\phi|^2 \geq 0$ + $\sigma > 0$, Thus, integral is zero iff $\phi(x) \equiv 0$ (not e.f.) or $\lambda - \bar{\lambda} = 0 \Rightarrow \lambda$ real

Thus, <u>e.v. are real.</u>

<u>Unique e.f.'s (regular + sing. cases)</u>

Spse ϕ_1 + ϕ_2 e.f.'s corresponding to λ.

$$L(\phi_1) + \lambda \sigma \phi_1 = 0$$
$$L(\phi_2) + \lambda \sigma \phi_2 = 0$$

$\therefore$ since λ same

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = 0$$

Lagrange identity $\Rightarrow$

$$\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = \frac{d}{dx}\left[p\left(\phi_2 \frac{d\phi_1}{dx} - \phi_1 \frac{d\phi_2}{dx}\right)\right] = 0$$

$$\therefore \quad p\left(\phi_2 \frac{d\phi_1}{dx} - \phi_1 \frac{d\phi_2}{dx}\right) = \text{constant}$$

This constant is zero if we have regular SL Prob with at least one B.C. $= 0$

$$\Rightarrow \qquad \phi_1 \frac{d\phi_2}{dx} - \phi_2 \frac{d\phi_1}{dx} = 0 \quad \Leftrightarrow \quad \frac{d}{dx}\left(\phi_2/\phi_1\right) = 0$$

Thus, $\qquad \phi_2(x) = c\,\phi_1(x) \qquad$ with those b.c.

<u>Non unique</u> Already showed periodic b.c. give both sine & cosine, so not unique for given e.v. ($\lambda = 0$ is unique)

Nonuniqueness can create problems with orthogonality for a given e.v., but $\exists$ process Gram-Schmidt to create orthogonal set ($\lambda_m \neq \lambda_n$, always orthogonal)

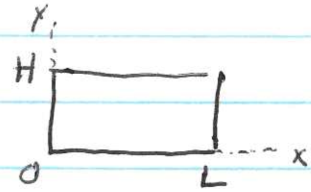
Chap 7 Higher Dimensional PDEs

Vibrating Membrane $\quad \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

u depending on t & 2-3 space var.

Heat Conduction $\quad \frac{\partial u}{\partial t} = k \nabla^2 u$

7.3 Rectangular Membrane



$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

B.C. $u(0,y,t)=0$, $u(x,0,t)=0$, $u(L,y,t)=0$, $u(x,H,t)=0$

I.C. $u(x,y,0)=\alpha(x,y) \qquad u_t(x,y,0)=\beta(x,y)$

Sep. Var Let $u(x,y,t) = h(t)\,\phi(x,y)$

$$h''\phi = c^2 h\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) \qquad h'' = -\lambda c^2 h$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\lambda\phi$$

$\phi(0,y)=0$, $\phi(x,0)=0$, $\phi(L,y)=0$, $\phi(x,H)=0$

Eqn. in ϕ is a linear homogeneous PDE

Let $\phi(x,y) = f(x)g(y)$ $\qquad$ ($u(x,y,t) = h(t)f(x)g(y)$]

could start

From ϕ PDE

$$f''g + fg'' = -\lambda fg \quad \text{or} \quad \frac{f''}{f} = -\lambda - \frac{g''}{g} = -\mu$$

Two SL-Prob.

(1) $\quad f'' + \mu f = 0 \quad$ w/ $f(0)=0$ & $f(L)=0$

(2) $\quad g'' + (\lambda-\mu)g = 0 \quad$ w/ $g(0)=0$ & $g(H)=0$

Solving (1), has e.v. $\mu_n = \left(\frac{n\pi}{L}\right)^2$, $n=1,2,\dots$

e.f. $f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

(2) also SL - e.v. prob with e.v.'s

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, \ldots$$

with e.f.'s $g_{nm}(y) = \sin\left(\frac{m\pi y}{H}\right)$

The separation constant is $\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$

Time soln

$$h_{nm}(t) = a_{nm} \cos\left(c\sqrt{\lambda_{nm}}\, t\right) + b_{nm} \sin\left(c\sqrt{\lambda_{nm}}\, t\right)$$

<u>Superposition Principle</u>

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \cos\left(c\sqrt{\lambda_{nm}}\, t\right)$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) \sin\left(c\sqrt{\lambda_{nm}}\, t\right)$$

Must satisfy I.C.'s

$$\alpha(x,y) = u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

Mult. by $\sin\left(\frac{j\pi x}{L}\right)$ & integrate 0 to L, Mult. by $\sin\left(\frac{k\pi y}{H}\right)$ & $\int_0^H \cdot\, dy$

$$a_{nm} = \frac{4}{LH} \int_0^H \int_0^L \alpha(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx\, dy$$

$$\beta(x,y) = u_t(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{nm}}\, b_{nm} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right)$$

$$b_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}} \int_0^H \int_0^L \beta(x,y) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{H}\right) dx\, dy$$

Vibrating Circular Membrane

PDE $\quad \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$

B.C. $\quad u(a, \theta, t) = 0$

I.C. $\quad u(r, \theta, 0) = \alpha(r, \theta)$, $\frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta)$

Separation of Variables

$$u(r, \theta, t) = h(t)\,\phi(r)\,g(\theta)$$

$$\frac{h''}{c^2 h} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{1}{r^2 g}\frac{d^2 g}{d\theta^2} = -\lambda, \text{ so } \quad h'' + \lambda c^2 h = 0$$

$$\frac{g''}{g} = -\frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) - \lambda r^2 = -\mu, \text{ so } \quad g'' + \mu g = 0 \ \&$$

$$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + (\lambda r^2 - \mu)\phi = 0$$

---

2 S-L Problems

1. $g'' + \mu g = 0$, $g(-\pi) = g(\pi)$, $g'(-\pi) = g'(\pi)$

$\mu = m^2 \qquad m = 0, 1, 2 \ldots$

e.f. $\quad g_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta) \qquad g_0(\theta) = 1$

2. $r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + (\lambda r^2 - m^2)\phi = 0$, $\phi(a) = 0$, $|\phi(0)|$ bdd

S-L form $\quad \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0 \quad$ (weighting factor $\sigma(r) = r$)

Singular S-L Prob. $p(r) = r$, $\sigma(r) = r$, $q(r) = -m^2/r$

with 1. $p(0) = 0$ and $\sigma(0) = 0$ (not positive)

2. B.C. at $r = 0$ not regular form

3. $q(r) \to \infty$ as $r \to 0$ (not cont.)

Still claim same prop. of reg. S-L Prob

1. infinitely many eiv. λ_{nm} w/ e.f. $\phi_{nm}(r)$

2. Orthogonality $\int_0^a \phi_{mn_1}(r)\, \phi_{mn_2}(r)\, r\, dr = 0$

with weighting fcn $\sigma(r) = r$.

Bessel's DE

Can write S-L Prob.

$$r^2 \frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + (\lambda r^2 - m^2)\phi = 0$$

Let $z = \sqrt{\lambda}\, r$

$$z^2 \frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\phi = 0$$

Bessel's d.e. of order m.

Singular Pt.

$$\frac{d^2\phi}{dz^2} + \frac{1}{z}\frac{d\phi}{dz} + \left(1 - \frac{m^2}{z^2}\right)\phi = 0$$

coef. are well-defined except at $z = 0$

Singular at $z = 0$, because $\frac{1}{z}$ & $\left(1 - \frac{m^2}{z^2}\right)$ undefined

Solns guaranteed ~~except~~ at any ordinary pt., but not $z = 0$.

Investigate near $z = 0$, z^2 small compared to m^2 (unless $m \sim 0$), so Bessel's d.e. becomes

$$z^2 \phi'' + z\phi' - m^2 \phi = 0$$

Try $\phi(z) = z^s$ & $s(s-1) + s - m^2 = s^2 - m^2 = 0$

This is the <u>indicial eqn.</u> (Method of Frobenius for power series solns. of sing. d.e.)

Soln. $\phi(z) \approx z^m \ \& \ z^{-m} \quad (m>0)$

For $m=0$ $\quad \phi(z) \approx 1 \ \& \ \ln(z)$

Show series solns Maple + Graphs

7.7.6 Asymptotic Properties

Bessel's DE. of order m

$$z^2 \frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2-m^2)\phi = 0$$

Has soln

$$\phi(z) = c_1 J_m(z) + c_2 Y_m(z)$$

$J_m(z)$ is Bessel's function of the first kind of order m
$Y_m(z)$ is Bessel's function of the second kind of order m

Asymptotically, as $z \to 0$

$$J_m(z) \sim \begin{cases} 1 & m=0 \\ \frac{1}{2^m m!} z^m & m>0 \end{cases}$$

$$Y_m(z) \sim \begin{cases} \frac{2}{\pi}\ln(z) & m=0 \\ -\frac{2^m (m-1)!}{\pi} z^{-m} & m>0 \end{cases}$$

$J_m(z)$ bdd as $z \to 0$ $\quad Y_n(z)$ unbdd as $z \to 0$.

Identities $\quad \frac{d}{dx}\left(x^{-\mu} J_\mu(x)\right) = -x^{-\mu} J_{\mu+1}(x)$

$$\frac{d}{dx}\left(x^{\mu} J_\mu(x)\right) = x^{\mu} J_{\mu-1}(x)$$

$$\int x^{\mu} J_\mu(x)\, x\, dx = x^{\mu} J_{\mu-1}(x)$$

E.V. Prob w/ Bessel's Fcns

$$\left.\begin{array}{l} \frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0 \\ \phi(a) = 0 \qquad |\phi(0)| < \infty \quad \text{bdd.} \end{array}\right\} \text{ sing. SL Prob.}$$

Change of variables $z = \sqrt{\lambda}\, r$ converts to

$$z^2 \frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - m^2)\phi = 0$$

$\therefore$ Soln is

$$\phi(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r)$$

Bddness $\Rightarrow c_2 = 0$

$$\therefore \phi(r) = c_1 J_m(\sqrt{\lambda} r)$$

$\phi(a) = 0$ gives eigenvalues. or $J_m(\sqrt{\lambda}\, a) = 0$

$J_m(z)$ has infinitely many zeroes. Let z_{mn} designate n^{th} zero of $J_m(z)$, then

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$$

The corresponding e.f.

$$\phi_{mn}(r) = J_m\left(z_{mn}\frac{r}{a}\right) \qquad m = 0, 1, 2, \ldots \quad n = 0, 1, 2, \ldots$$

e.g. $z_{01} = 2.40483$, $z_{02} = 5.52008$, $z_{03} = 8.65373$ $\sim \pi$ apart

Orthogonality

$$\int_0^a J_m(\sqrt{\lambda_{mp}}\, r) J_m(\sqrt{\lambda_{mq}}\, r)\, \underline{r}\, dr = 0 \qquad p \neq q$$

These e.f.'s form a complete set

Any piecewise smooth fcn

$$\alpha(r) \approx \sum_{n=1}^{\infty} a_n J_m(\sqrt{\lambda_{mn}}\, r)$$ Fourier-Bessel series

with
$$a_n = \frac{\int_0^a \alpha(r) J_m(\sqrt{\lambda_{mn}}\, r)\, r\, dr}{\int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\, dr}$$

IVP for Vibrating Circular Membrane

PDE $\quad \dfrac{\partial^2 u}{\partial t^2} = c^2\left(\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}\right)$ $\quad -\pi < \theta \le \pi$, $\ 0 \le r < a$

B.C. $\quad u(a,\theta,t) = 0 \quad \begin{cases} u(r,-\pi,t) = u(r,\pi,t) \\ \dfrac{\partial u}{\partial \theta}(r,-\pi,t) = \dfrac{\partial u}{\partial \theta}(r,\pi,t) \end{cases}$

I.C. $\quad u(r,\theta,0) = \alpha(r,\theta) \qquad$ simplicity $\dfrac{\partial u}{\partial t}(r,\theta,0) = 0$

Sep. Var

$u(r,\theta,t) = h(t)\,\phi(r)\,g(\theta)$

t-equ. $\quad h'' + \lambda c^2 h = 0 \quad \longrightarrow \quad h(t) = c_1\cos(c\sqrt{\lambda}\, t) + c_2 \sin(c\sqrt{\lambda}\, t)$

θ-equ. $\quad g'' + \mu g = 0 \quad$ with $\ g(-\pi) = g(\pi)\ \ \&\ \ g'(-\pi) = g'(\pi)$

$g_0(\theta) = 1 \qquad g_m(\theta) = c_1\cos(m\theta) + c_2\sin(m\theta)$

r-equ. $\quad \dfrac{d}{dr}\left(r\dfrac{d\phi}{dr}\right) + \left(\lambda r - \dfrac{m^2}{r}\right)\phi = 0\ , \quad \phi(a) = 0 \quad |\phi(0)| < \infty.$

$\phi_{mn}(r) = J_m(\sqrt{\lambda_{mn}}\, r)$

If $\dfrac{\partial u}{\partial t}(r,\theta,0) = 0$, can omit $\sin(c\sqrt{\lambda}\, t)$ terms

~~Superposition gives~~

$\cancel{u(r,\theta,t) =}$

Bessel's d.e.

$$z^2\phi'' + z\phi' + (z^2 - m^2)\phi = 0$$

$$\cancel{\phi(z) = \sum_{n=0}^{\infty} a_n z^n \quad \phi' = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad \phi'' = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}}$$

$$\cancel{k = n+2}$$

$$\cancel{\sum_{n=2}^{\infty} n(n-1) a_n z^n + \sum_{n=1}^{\infty} n a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+2} - m^2 \sum_{n=0}^{\infty} a_n z^n}$$

$$\cancel{\sum_{n=2}^{\infty} n(n-1) a_n z^n + \sum_{n=1}^{\infty} n a_n z^n + \sum_{n=2}^{\infty} a_{n-2} z^n - m^2 \sum_{n=0}^{\infty} a_n z^n = 0}$$

$$\cancel{\sum_{n=2}^{\infty} \left[(n^2 - m^2) a_n + a_{n-2}\right] z^n - m^2 a_0 + a_1 (1 - m^2) z = 0}$$

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^{r+n} \qquad \phi'(z) = \sum_{n=0}^{\infty} (r+n) a_n z^{r+n-1} \qquad \cancel{\phi''(z) = \sum_{n=0}^{\infty}}$$

$$\phi''(z) = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n z^{r+n-2}$$

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n z^{r+n} + \sum_{n=0}^{\infty} (r+n) a_n z^{r+n} - m^2 \sum_{n=0}^{\infty} a_n z^{r+n} \qquad k = n+2$$

$$\begin{cases} + \sum_{n=0}^{\infty} a_n z^{r+n+2} = 0 \\ + \sum_{n=2}^{\infty} a_{n-2} z^{r+n+2} = 0 \end{cases}$$

$n = 0$: $r^2 - m^2 = 0$ indicial equ $r = \pm m$

Spse $m = 0$ repeat root $r_{1,2} = 0$ a_0 arb $a_1 = 0$

Recurrance relation

$$a_n = \frac{-a_{n-2}}{n^2}, \qquad a_2 = -\frac{a_0}{4} = -\frac{a_0}{2^2}$$

$$a_4 = \frac{a_0}{2^4 2^2} \quad \cdots \quad a_{2k} = \frac{(-1)^k a_0}{2^{2k}(k!)^2}$$

$$J_0(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k}(k!)^2} \qquad z > 0$$

Super position Principle gives

$$u(r,\theta,t) = \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}}\, r) \cos(c\sqrt{\lambda_{0n}}\, t)$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)\right) J_m(\sqrt{\lambda_{mn}}\, r) \cos(c\sqrt{\lambda_{mn}}\, t)$$

From I.C.

$$\alpha(r,\theta) = \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}}\, r)$$

$$+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(A_{mn} \cos(m\theta) + B_{mn} \sin(m\theta)\right) J_m(\sqrt{\lambda_{mn}}\, r)$$

Std Fourier series in θ, Fourier-Bessel for r

use orthogonality and obtain

$$A_{0n} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) J_0(\sqrt{\lambda_{0n}}\, r)\, r\, dr\, d\theta}{2\pi \int_0^a J_0^2(\sqrt{\lambda_{0n}}\, r)\, r\, dr}$$

$$A_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) \cos(m\theta) J_m(\sqrt{\lambda_{mn}}\, r)\, r\, dr\, d\theta}{\pi \int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\, dr}$$

$$B_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta) \sin(m\theta) J_m(\sqrt{\lambda_{mn}}\, r)\, r\, dr\, d\theta}{\pi \int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\, dr}$$

Easier notation

$$\alpha(r,\theta) = \sum_{\lambda} A_\lambda \phi_\lambda(r,\theta)$$

$$A_\lambda = \frac{\iint \alpha(r,\theta)\, \phi_\lambda(r,\theta)\, dA}{\iint \phi_\lambda^2(r,\theta)\, dA} \qquad dA = r\, dr\, d\theta$$

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Circularly Symmetric Case

$u \doteq u(r,t)$

PDE $\quad \dfrac{\partial^2 u}{\partial t^2} = \dfrac{c^2}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$

B.C. $\quad u(a,t)=0 \qquad (\,|u(0)|<\infty)$

I.C. $\quad u(r,0) = \alpha(r) \qquad \dfrac{\partial u}{\partial t}(r,0) = \beta(r)$

Sep. Var. $\quad u(r,t) = \phi(r)\,h(t)$

$$\frac{h''}{c^2 h} = \frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda$$

t-eqn $\quad h'' + \lambda c^2 h = 0$

SL Prob (sing.)

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r\phi = 0 \qquad \phi(a)=0 \qquad |\phi(0)|<\infty$$

Soln. with $m=0$ of Bessel's Eqn.

$$\phi(r) = c_1 J_0(\sqrt{\lambda}\,r) + c_2 Y_0(\sqrt{\lambda}\,r) \qquad \text{bdd} \Rightarrow c_2 = 0$$

e.v. satisfy λ_n such that $J_0(\sqrt{\lambda_n}\,a) = 0$

e.f. $\quad \phi_n(r) = J_0(\sqrt{\lambda_n}\,r)$

$$h(t) = a_n\cos(c\sqrt{\lambda_n}\,t) + b_n\sin(c\sqrt{\lambda_n}\,t)$$

$$u(r,t) = \sum_{n=1}^{\infty}\left(a_n\cos(c\sqrt{\lambda_n}\,t) + b_n\sin(c\sqrt{\lambda_n}\,t)\right) J_0(\sqrt{\lambda_n}\,r)$$

I.C.

$$u(r,0)=\alpha(r)=\sum_{n=1}^{\infty} a_n J_0(\sqrt{\lambda_n}\, r)$$

$$a_n = \frac{\int_0^a \alpha(r) J_0(\sqrt{\lambda_n}\, r)\, r\, dr}{\int_0^a J_0^2(\sqrt{\lambda_n}\, r)\, r\, dr}$$

$$u_t(r,0)=\beta(r)=\sum_{n=1}^{\infty} b_n c\sqrt{\lambda_n}\; J_0(\sqrt{\lambda_n}\, r)$$

$$b_n = \frac{\int_0^a \beta(r) J_0(\sqrt{\lambda_n}\, r)\, r\, dr}{c\sqrt{\lambda_n}\int_0^a J_0^2(\sqrt{\lambda_n}\, r)\, r\, dr}$$

More on Bessel Functions

$$z^2 \phi'' + z\phi' + (z^2 - m^2)\phi = 0$$

Bessel's Eqn

$$\frac{d^2\phi}{dz^2} = -\left(1 - \frac{m^2}{z^2}\right)\phi - \frac{1}{z}\frac{d\phi}{dz}$$

Compare to spring-mass

$$\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt}$$

1. Like time varying frictional force ($c \sim 1/t$) getting weaker with time (less than exp. decay)
2. Restoring force ($k \sim (1 - \frac{m^2}{t^2})$) approaches constant oscillation

Small z

$J_0(z) \approx 1$	$Y_0(z) \approx \frac{2}{\pi} \ln(z)$
$J_1(z) \approx \frac{1}{2} z$	$Y_1(z) \approx -\frac{2}{\pi} z^{-1}$
$J_2(z) \approx \frac{1}{8} z^2$	$Y_2(z) \approx -\frac{4}{\pi} z^{-2}$

Large z

$$\left.\begin{aligned} J_m(z) &\sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \\ Y_m(z) &\sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right) \end{aligned}\right\} \text{ as } z \to \infty$$

Zeroes asymptotically separated by π

Bessel - Nodal Curves - See R.C. Matlab + Maple

Series Representation - Did Maple, shown in text

Laplace's Eqn. in a Circular Cylinder

$$\nabla^2 u = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\left.\begin{array}{ll} 1. & u(r,\theta,H) = \beta(r,\theta) \\ 2. & u(r,\theta,0) = \alpha(r,\theta) \\ 3. & u(a,\theta,z) = \gamma(\theta,z) \end{array}\right\} \text{Solve with any 1 B.C. nonzero; 3 problems}$$

3 other B.C. periodicity & bddness

Sep. of Var $\quad u(r,\theta,z) = \phi(r)\, g(\theta)\, h(z)$

$$\frac{1}{r\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \frac{1}{r^2 g} g'' + \frac{h''}{h} = 0$$

Don't expect oscillations in z for B.C. 1+2

z - dependence $\quad \frac{h''}{h} = \lambda \quad$ or $\quad h'' - \lambda h = 0$

$$\therefore \quad \frac{r}{\phi}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda r^2 = -\frac{g''}{g} = \mu$$

For all problems, have SL - Prob in θ

$$g'' + \mu g = 0 \qquad g(-\pi) = g(\pi) \qquad g'(-\pi) = g'(\pi)$$

$$\mu_m = m^2, \quad m = 0, 1, 2 \ldots$$

$$g_0(\theta) = 1 \qquad g_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta)$$

Remaining d.e.'s

$$h'' - \lambda h = 0$$

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0.$$

Problem 1.

B.C. $\quad u(a,\theta,z) = 0, \quad u(r,\theta,0) = 0$

$\quad u(r,\theta,H) = \beta(r,\theta)$

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2nd SL - Problem

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0 \qquad \phi(a) = 0\ ,\ |\phi(0)| < \infty$$

$$\phi(r) = c_1 J_m(\sqrt{\lambda}\, r) + c_2 Y_m(\sqrt{\lambda}\, r) \qquad \text{bdd} \Rightarrow c_2 = 0$$

e.v. λ_{mn} satisfy $J_m(\sqrt{\lambda_{mn}}\, a) = 0$

$$\phi_{mn}(r) = J_m(\sqrt{\lambda_{mn}}\, r)$$

$$h'' - \lambda_{mn} h = 0 \qquad (\lambda_{mn} > 0)$$

$$h(z) = c_1 \cosh(\sqrt{\lambda_{mn}}\, z) + c_2 \sinh(\sqrt{\lambda_{mn}}\, z)$$

$$h(0) = 0 \Rightarrow c_1 = 0$$

$$h(z) = \sinh(\sqrt{\lambda_{mn}}\, z)$$

Superposition

$$u(r,\theta,z) = \sum_{n=1}^{\infty} A_{0n} \sinh(\sqrt{\lambda_{0n}}\, z)\, J_0(\sqrt{\lambda_{0n}}\, r)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right) J_m(\sqrt{\lambda_{mn}}\, r) \sinh(\sqrt{\lambda_{mn}}\, z)$$

e.g. $A_{mn} = \dfrac{\int_{-\pi}^{\pi}\int_0^a \beta(r,\theta)\cos(m\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)\, r\, dr\, d\theta}{\pi \sinh(\sqrt{\lambda_{mn}}\, H)\int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\, dr}$ $\neq 0$

Appropriate Fourier coef.

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Similar for Prob 2

B.C. $u(a,\theta,z) = 0 \qquad u(r,\theta,H) = 0$

$$u(r,\theta,0) = \alpha(r,\theta)$$

$$u(r,\theta,z) = \sum_{n=1}^{\infty} A_{0n} \sinh(\sqrt{\lambda_{0n}}\,(H-z))\, J_0(\sqrt{\lambda_{0n}}\, r)$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right) J_m(\sqrt{\lambda_{mn}}\, r) \sinh(\sqrt{\lambda_{mn}}\,(H-z))$$

e.g. $$B_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^a \alpha(r,\theta)\sin(m\theta)\, J_m(\sqrt{\lambda_{mn}}\, r)\, r\, dr\, d\theta}{\pi \sinh(\sqrt{\lambda_{mn}}\, H)\int_0^a J_m^2(\sqrt{\lambda_{mn}}\, r)\, r\, dr}$$

Prob 3

B.C. $u(r,\theta,0) = 0$ & $u(r,\theta,H) = 0$

$u(a,\theta,z) = \gamma(\theta,z)$

2nd SL - Prob is

$$h'' - \lambda h = 0 \quad w/ \quad h(0) = 0 \;,\; h(H) = 0$$

From before

$$\lambda = -\left(\frac{n\pi}{H}\right)^2 \qquad n = 1, 2, \ldots$$

$$h(z) = \sin \frac{n\pi z}{H}$$

θ as before (1st SL Prob)

r - Eqn becomes

$$\frac{d}{dr}\left(r \frac{d\phi}{dr}\right) + \left(-\left(\frac{n\pi}{H}\right)^2 r - \frac{m^2}{r}\right)\phi = 0$$

still have b.c. $|\phi(0)| < \infty$.

This eqn similar to Bessel's Eqn., but the sign is wrong.

If we let $s = i\left(\frac{n\pi}{H}\right) r$, transform to

$$s^2 \phi'' + s\phi' + (s^2 - m^2)\phi = 0$$

or $\phi(s) = c_1 J_m(s) + c_2 Y(s) = c_1 J_m\left(i \frac{n\pi r}{H}\right) + c_2 Y_m\left(i \frac{n\pi r}{H}\right)$

which gives imaginary Bessel fcns - NOT useful

Let $w = \frac{n\pi}{H} r$, obtain

$$w^2 \frac{d\phi}{dw} + w \frac{d\phi}{dw} + (-w^2 - m^2)\phi = 0$$

Solns are modified Bessel's fcns (sing. at $w=0$)

$$\phi(r) = c_1 K_m\left(\frac{n\pi}{H} r\right) + c_2 I_m\left(\frac{n\pi r}{H}\right)$$

modified Bessel fcn of order m of the 2nd kind — modified Bessel fcn of order m of the 1st kind

$$\lim_{r \to 0} K_m\left(\frac{n\pi r}{H}\right) = \infty \implies c_1 = 0$$

Superposition Principle

$$u(r,\theta,z) = \sum_{n=1}^{\infty} E_{0n}\, I_m\left(\frac{n\pi r}{H}\right) \sin\left(\frac{n\pi z}{H}\right)$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(E_{mn}\cos(m\theta) + F_{mn}\sin(m\theta)\right) I_m\left(\frac{n\pi r}{H}\right) \sin\left(\frac{n\pi z}{H}\right)$$

Provided $I_m\left(\frac{n\pi a}{H}\right) \neq 0$, Fourier coef. readily found

$$E_{0n} = \frac{\int_{-\pi}^{\pi}\int_{0}^{H} \gamma(\theta,z)\sin\left(\frac{n\pi z}{H}\right) dz\, d\theta}{\pi H\, I_m\left(\frac{n\pi a}{H}\right)}$$

$\int_0^H \sin^2\left(\frac{n\pi z}{H}\right) dz = \frac{H}{2}$

$\int_{-\pi}^{\pi} 1\, d\theta = 2\pi$

$$E_{mn} = \frac{2\int_{-\pi}^{\pi}\int_{0}^{H} \gamma(\theta,z)\cos(m\theta)\sin\left(\frac{n\pi z}{H}\right) dz\, d\theta}{\pi H\, I_m\left(\frac{n\pi a}{H}\right)}$$

$$F_{mn} = \frac{2\int_{-\pi}^{\pi}\int_{0}^{H} \gamma(\theta,z)\sin(m\theta)\sin\left(\frac{n\pi z}{H}\right) dz\, d\theta}{\pi H\, I_m\left(\frac{n\pi a}{H}\right)}$$

Properties of Modified Bessel Fcns

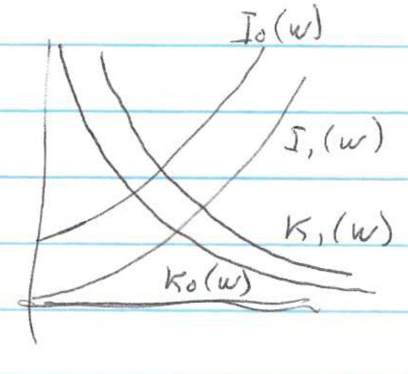
$$K_m(w) \sim \sqrt{\frac{\pi}{2}}\, \frac{e^{-w}}{w^{1/2}} \quad \text{as } w \to \infty$$

$$K_m(w) \sim \begin{cases} \ln(w) & m=0 \\ \frac{1}{2}(m-1)!\left(\frac{1}{2}w\right)^{-m} & m\neq 0 \end{cases} \quad \text{as } w \to 0$$

Sing. ex $m=0$

$$I_m(w) \sim \sqrt{\frac{1}{2\pi w}}\, e^{w} \quad \text{as } w \to \infty$$

$$I_m(w) \sim \frac{1}{m!}\left(\frac{1}{2}w\right)^{m} \quad \text{as } w \to 0$$



Spherical Problems & Legendre Polynomials

Vibrations inside Earth (

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

Separation of Variables

$$u(\rho, \theta, \phi, t) = w(\rho, \theta, \phi)\, h(t)$$

Obtain

$$h'' + \lambda c^2 h = 0 \qquad \& \qquad \nabla^2 w + \lambda w = 0$$

In spherical coordinates

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial w}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 w}{\partial \theta^2} + \lambda w = 0$$

Sep. Var.

$$w(\rho, \theta, \phi) = f(\rho)\, q(\theta)\, g(\phi)$$

Notice that none of the coef. depend on θ

$$\frac{qg}{\rho^2}\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \frac{fq}{\rho^2 \sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \frac{fg}{\rho^2 \sin^2\phi} q'' + \lambda fqg = 0$$

$$\frac{\sin^2\phi}{f}\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \frac{\sin\phi}{g}\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \lambda\rho^2\sin^2\phi = -\frac{q''}{q} = +\mu$$

1st SL-Prob with periodicity $q'' + \mu^2 q = 0 \qquad q(-\pi) = q(\pi),\ q'(-\pi) = q'(\pi)$

$\mu_0 = 0,\ q_0(\theta) = 1 \qquad \mu_m = m \qquad q_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta)$

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$$\frac{1}{f}\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \lambda\rho^2 = -\frac{1}{g\sin\phi}\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \frac{m^2}{\sin^2\phi} = \mu$$

Two ODE in ρ + ϕ

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda\rho^2 - \mu) f = 0$$

$$\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) + \left(\mu \sin\phi - \frac{m^2}{\sin\phi}\right) g = 0$$

Note that the ϕ eqn. doesn't depend on both λ + μ
However, this eqn. is singular at $\phi = 0$ + π (N + S poles)
Clearly, not a regular SL-Prob., also no b.c.'s (physical)
Impose bddness $|g(0)| < \infty$ + $|g(\pi)| < \infty$
Obtain e.v.'s μ_{nm} for each m + e.f.'s $g_{nm}(\phi)$
orthogonal w/ weighting fcn. $\sin\phi$

The ρ eqn. has e.v. λ + weighting fcn ρ^2
$f(a) = 0$ + $|f(0)| < \infty$ provided homo b.c. + this
sing prob.

<u>Associated Legendre Fcns + Legendre Poly.</u>

Let $x = \cos\phi$ (non-obvious change)

Since $\phi \in [0, \pi] \implies x \in [-1, 1]$

$$\frac{dg}{d\phi} = \frac{dx}{d\phi}\frac{dg}{dx} = -\sin\phi \frac{dg}{dx}, \qquad \frac{d(dg/dx)}{d\phi} = \frac{dx}{d\phi}\frac{d(dg/dx)}{dx} = -\sin\phi \frac{d^2 g}{dx^2}$$

$$\frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\phi}\right) = \frac{d}{d\phi}\left(-\sin^2\phi \frac{dg}{dx}\right) = -\sin^2\phi \frac{d}{d\phi}\left(\frac{dg}{dx}\right) - 2\sin\phi\cos\phi \frac{dg}{dx}$$

$$= \sin^3\phi \frac{d^2 g}{dx^2} - 2\sin\phi\cos\phi \frac{dg}{dx}$$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dg}{d\phi}\right) + \left(\mu - \frac{m^2}{\sin^2\phi}\right)g = 0$$

Becomes

$$\sin^2\phi\,\frac{d^2g}{dx^2} - 2\cos\phi\,\frac{dg}{dx} + \left(\mu - \frac{m^2}{\sin^2\phi}\right)g = 0$$

$\sin^2\phi = 1 - x^2$

Legendre Eqn

$$(1-x^2)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + \left(\mu - \frac{m^2}{1-x^2}\right)g = 0$$

In SL form

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right) + \left(\mu - \frac{m^2}{1-x^2}\right)g = 0 \qquad -1 < x < 1$$

This is sing. at $x=-1$ and $x=1$. These are regular singular pts. (N + S poles)

What happens near $x = \pm 1$?

For $x \approx 1$, $1 - x^2 = (1-x)(1+x) \approx 2(1-x)$

$\therefore$ approx. eqn. becomes small

$$-2\frac{d}{dx}\left[(x-1)\frac{dg}{dx}\right] + \left(\cancel{\mu} + \frac{m^2}{2(x-1)}\right)g = 0$$

Try $g(x) = (x-1)^p$

$$-2\frac{d}{dx}\left[(x-1)p(x-1)^{p-1}\right] + \frac{m^2}{2}(x-1)^{p-1} = 0$$

$$-2p^2(x-1)^{p-1} + \frac{m^2}{2}(x-1)^{p-1} = 0$$

Char. Eqn $p^2 - \frac{m^2}{4} = 0 \Rightarrow p = \pm\frac{m}{2}$

$$\therefore g(x) \approx a(x-1)^{m/2} + b(x-1)^{-m/2}$$

Want bdd soln, so take $g(x) \approx a(x-1)^{m/2}$ near $x=1$

Bdd soln.

$$g(x) = P_n^m(x)$$

<u>If $m=0$: Legendre Polynomials</u>

$$\frac{d}{dx}\left[(1-x^2)\frac{dg}{dx}\right] + n(n+1)g = 0$$

$P_0(x) = 1$, $P_1(x) = x = \cos(\phi)$, $P_2 = \frac{1}{2}(3x^2-1) = \frac{1}{4}(3\cos(2\phi)+1)$

$x=1$, $\phi=0$ is North pole.

<u>Rodrigues' Formula</u> - finding general n^{th} order Legendre Polynomial

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Legendre Polynomials are orthogonal w/ weight 1

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \qquad n \neq m$$

Can be obtained by Gram-Schmidt process

Form a complete set of polynomials

Only e.v.'s $\mu_n = n(n+1)$

<u>If $m>0$, Associated Legendre fcns</u>

For $m>0$, the e.v.'s are basically the same as $m=0$.

$$g(x) = P_n^m(x) = (x^2-1)^{m/2}\frac{d^m}{dx^m}P_n(x), \qquad n \geq m$$

where $P_n(x)$ is n^{th} degree Legendre Poly.

e.v.'s satisfy

$$\mu = n(n+1) \qquad \text{with} \qquad n \geq m$$

Problems arise

1. Soln. bdd at $x=1$ is usually unbdd at $x=-1$
   (think $x=-1$ needing linear combination above.
2. Few values of μ allow bdd soln at $x=1$ AND $x=-1$
3. Proof requires power series methods (Method of Frobenious)
   omit proof
4. If $\mu = n(n+1)$, then one series soln. is a polynomial, so
   bdd at $x = \pm 1$.

Consider

$$(1-x^2)\frac{d^2 g}{dx^2} - 2x\frac{dg}{dx} + \mu g = 0$$

use k for index

Let $g(x) = \sum_{k=0}^{\infty} a_n x^k$, $g'(x) = \sum_{k=1}^{\infty} k a_n x^{k-1}$, $g''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

$$(1-x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + \mu \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)a_k x^k - 2\sum_{k=1}^{\infty} k a_k x^k + \mu\sum_{k=0}^{\infty} a_k x^k = 0$$

$k=0$: $2a_2 + \mu a_0 = 0$ $\qquad a_2 = -\frac{\mu a_0}{2}$

$k=1$: $3\cdot 2 a_3 - 2a_1 + \mu a_1 = 0$ $\qquad a_3 = \frac{2-\mu}{3!} a_1$

$k \geq 2$: Recurrence Relation

$$a_{k+2} = \frac{k(k+1) - \mu}{(k+2)(k+1)} a_k \qquad \mu = n(n+1)$$

one series termintes at $k = n$

Thus, the d.e.

$$\frac{d}{dx}\left[(1-x^2)\frac{dg}{dx}\right] + \left(\mu - \frac{m^2}{1-x^2}\right)g = 0 \qquad \mu = n(n+1)$$

has two indep. solns associated Legendre fcns (spherical harmonics)
first kind $P_n^m(x)$ (bdd $x=\pm 1$) + second kind $Q_n^m(x)$

Radial E.V. Problem

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With $\mu = n(n+1)$, radial SL-Prob

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \left(\lambda \rho^2 - n(n+1)\right) f = 0$$

with $n \geq m$ for fixed m.

B.C. $f(a) = 0$, $|f(0)| < \infty$

Not quite Bessel's Eqn.

Soln given by

$$f(\rho) = \frac{1}{\sqrt{\rho}} J_{n+1/2}(\sqrt{\lambda}\,\rho)$$ spherical Bessel fcn

E.V. found by solving

$$J_{n+1/2}(\sqrt{\lambda}\, a) = 0$$ λ_{nk} infinitely many

Zeroes of Bessel fcns of order $n+1/2$

Spherical Bessel fcns are related to trig. fcns.

$$\frac{1}{\sqrt{x}} J_{n+1/2}(x) = x^n \left(-\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin(x)}{x}\right)$$

Vibration in sphere $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$

λ_{nk}

$$u(\rho, \theta, \phi, t) = \sum_k \sum_n \sum_m \begin{Bmatrix} \cos(c\sqrt{\lambda}t) \\ \sin(c\sqrt{\lambda}t) \end{Bmatrix} \frac{J_{n+1/2}(\sqrt{\lambda}\rho)}{\sqrt{\rho}} \begin{Bmatrix} \cos(m\theta) \\ \sin(m\theta) \end{Bmatrix} P_n^m(\cos\phi)$$

weighting fcn $(\rho^2 \sin\phi)\ d\rho\, d\theta\, d\phi$

7.10.1c $u(\rho, \theta, \phi, 0) = F(\rho, \phi)$ $\frac{\partial u}{\partial t}(\rho, \theta, \phi, 0) = 0$

$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}$

Laplace Eqn in Spherical Cavity

$$\nabla^2 u = 0$$

B.C. $u(a, \theta, \phi) = F(\theta, \phi)$ $\qquad u(\rho, \theta, \phi) = f(\rho) q(\theta) g(\phi)$

Same θ + ϕ eqns from above

$$\therefore q(\theta) = \begin{cases} \cos(m\theta) & m = 0, 1, \dots \\ \sin(m\theta) & m = 1, 2, \dots \end{cases}$$

$$g(\phi) = P_n^m(\cos\phi) \qquad n \geq m$$

Radial Eqn doesn't have e.v.

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) - n(n+1) f = 0$$

Let $f = \rho^r$, $\quad f' = r\rho^{r-1}$

$$r(r+1) - n(n+1) = r^2 + r - n(n+1) = 0$$

$$r = \frac{-1 \pm \sqrt{1 + 4n + 4n^2}}{2} = \frac{-1 \pm (1+2n)}{2} = n, -1-n$$

$|u(0, \theta, \phi)| < \infty \Rightarrow r = n$

$$f(\rho) = \rho^n$$

$$u(\rho, \theta, \phi) = \sum_{n=0}^{\infty} A_{0n} P_n(\cos\phi) \rho^n$$

$$+ \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \rho^n \left[A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right] P_n^m(\cos\phi)$$

B.C. $\quad F(\theta, \phi) = \sum_{n=0}^{\infty} a^n A_{0n} P_n(\cos\phi)$

$$+ \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} a^n \left\{A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)\right\} P_n^m(\cos\phi)$$

$$a^n B_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^{\pi} F(\theta, \phi)\sin(m\theta) P_n^m(\cos\phi)\sin(\phi)\, d\phi\, d\theta}{\int_{-\pi}^{\pi}\int_0^{\pi} \sin^2(m\theta)\left[P_n^m(\cos\phi)\right]^2 \sin(\phi)\, d\phi\, d\theta}$$

Similar A_{mn}