66 pts

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Math 342A Assignment 3

Problems, Section 11.3

Express each of the following integrals as a Γ function. By computer, evaluate numerically both the Γ function and the original integral.

9.

$$\int_0^\infty e^{-x^4}\,dx$$

Hint: Put $x^4 = u$.

We have the Gamma function Γ, defined

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx \tag{1}$$

such that for positive integers p we have

$$\Gamma(p) = (p-1)!. \tag{2}$$

Γ also has a recursive property:

$$\Gamma(p+1) = p\Gamma(p). \tag{3}$$

Let $u = x^4$. Then $du = 4x^3dx$ and $x = u^{1/4}$. Thus,

$$\int_0^\infty e^{-x^4}\,dx = \frac{1}{4}\int_0^\infty x^{-3}e^{-u}\,du = \frac{1}{4}\int_0^\infty u^{-3/4}e^{-u}\,du = \frac{1}{4}\int_0^\infty u^{1/4-1}e^{-u}\,du. \tag{4}$$

The integral in (4) is exactly the same as the integral in (1) for $p = \frac{1}{4}$, so we have

$$\int_0^\infty e^{-x^4}\,dx = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{1}{4}+1\right) = \Gamma\left(\frac{5}{4}\right). \tag{5}$$

`Maple` solves the integral in question as

$$\int_0^\infty e^{-x^4}\,dx = \frac{1}{4}\frac{\pi\sqrt{2}}{\Gamma(3/4)}, \tag{6}$$

and MATLAB numerically computes both (5) and (6) as 0.906402477055477. Therefore,

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$$\int_0^\infty e^{-x^4}\,dx = \Gamma\left(\frac{5}{4}\right) \approx 0.906402477055477. \tag{7}$$

10.

$$\int_0^\infty x^{-2/5}e^{-x}\,dx$$

From (1), we see that

$$\int_0^\infty x^{-2/5}e^{-x}\,dx = \int_0^\infty x^{3/5-1}e^{-x}\,dx = \Gamma\left(\frac{3}{5}\right). \tag{8}$$

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`Maple` confirms this exactly, and MATLAB computes (8) as 1.489192248812817. Thus,

$$\int_0^\infty x^{-2/5}\mathrm{e}^{-x}\mathrm{d}x = \Gamma\left(\frac{3}{5}\right) \approx 1.489192248812817. \tag{9}$$

11.

$$\int_0^\infty x^5\mathrm{e}^{-x^2}\mathrm{d}x$$

Hint: Put $x^2 = u$.

`Maple` simplifies this integral to 1. Let $u = x^2$. Then $\mathrm{d}u = 2x\mathrm{d}x$ and $x = u^{1/2}$. Thus,

$$\int_0^\infty x^5e^{-x^2}\mathrm{d}x = \frac{1}{2}\int_0^\infty x^4\mathrm{e}^{-u}\mathrm{d}u = \frac{1}{2}\int_0^\infty u^2\mathrm{e}^{-u}\mathrm{d}u = \frac{1}{2}\int_0^\infty u^{3-1}\mathrm{e}^{-u}\mathrm{d}u. \tag{10}$$

The integral in (10) is exactly (1) for $p = 3$, so we have

$$\int_0^\infty x^5\mathrm{e}^{-x^2}\mathrm{d}x = \frac{1}{2}\Gamma(3) = \frac{1}{2}2! = 1. \tag{11}$$

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12.

$$\int_0^\infty x\mathrm{e}^{-x^3}\mathrm{d}x$$

According to `Maple` ,

$$\int_0^\infty x\mathrm{e}^{-x^3}\mathrm{d}x = \frac{1}{3}\Gamma\left(\frac{2}{3}\right), \tag{12}$$

which MATLAB calculates to be 0.451372646475467. Now we compute the integral analytically. Let $u = x^3$. Then $\mathrm{d}u = 3x^2\mathrm{d}x$ and $x = u^{1/3}$. So we have

$$\int_0^\infty x\mathrm{e}^{-x^3}\mathrm{d}x = \frac{1}{3}\int_0^\infty x^{-1}\mathrm{e}^{-u}\mathrm{d}u = \frac{1}{3}\int_0^\infty u^{-1/3}\mathrm{e}^{-u}\mathrm{d}uv = \frac{1}{3}\int_0^\infty u^{2/3-1}\mathrm{e}^{-u}\mathrm{d}u. \tag{13}$$

Then from (1) we see that

$$\frac{1}{3}\int_0^\infty u^{2/3-1}\mathrm{e}^{-u}\mathrm{d}u = \frac{1}{3}\Gamma\left(\frac{2}{3}\right). \tag{14}$$

This is consistent with the answer given by `Maple` . Therefore, we have

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$$\int_0^\infty x\mathrm{e}^{-x^3}\mathrm{d}x = \frac{1}{3}\Gamma\left(\frac{2}{3}\right) \approx 0.451372646475467. \tag{15}$$

PROBLEMS, SECTION 11.7

Express the following integrals as B functions, and then, by (7.1), in terms of Γ functions. When possible, use Γ function formulas to write an exact answer in terms of π, √2, etc. Compare your answers with computer results and reconcile any discrepancies.

2.

$$\int_0^{\pi/2} \sqrt{\sin^3(x)\cos(x)}\,dx$$

We have

$$\int_0^{\pi/2} \sqrt{\sin^3(x)\cos(x)}\,dx = \int_0^{\pi/2} \left(\sin^3(x)\cos(x)\right)^{1/2} dx = \int_0^{\pi/2} \sin^{3/2}(x)\cos^{1/2}(x)\,dx. \tag{16}$$

The B function is defined in one way by

$$B(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta, \tag{17}$$

and in another as

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{18}$$

From (16) and (17) (letting $p = 5/4$ and $q = 3/4$) it follows that

$$\int_0^{\pi/2} \sqrt{\sin^3(x)\cos(x)}\,dx = \frac{1}{2}B\left(\frac{5}{4},\frac{3}{4}\right). \tag{19}$$

Then, from (18) we also get

$$\begin{aligned}\int_0^{\pi/2} \sqrt{\sin^3(x)\cos(x)}\,dx &= \frac{1}{2}\frac{\Gamma(5/4)\Gamma(3/4)}{\Gamma(5/4+3/4)} = \frac{1}{2}\frac{\Gamma(5/4)\Gamma(3/4)}{\Gamma(2)} \\ &= \frac{1}{2}\frac{\Gamma(5/4)\Gamma(3/4)}{1!} = \frac{1}{2}\Gamma\left(\frac{1}{4}+1\right)\Gamma\left(\frac{3}{4}\right) \\ &= \frac{1}{2}\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right).\end{aligned} \tag{20}$$

One known Γ function identity is

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin\pi p}. \tag{21}$$

From (20) and (21) we get

$$\int_0^{\pi/2} \sqrt{\sin^3(x)\cos(x)}\,dx = \frac{\pi}{8\sin\pi/4} = \frac{\pi}{8\left(\sqrt{2}/2\right)} = \frac{\pi}{4\sqrt{2}}. \tag{22}$$

`Maple` cannot compute the integral in question nicely. Instead, it gives

```
(1/2*I)*EllipticK((1/2)*sqrt(2))-(1/4*I)*EllipticPi(1/2-1/2*I,
(1/2)*sqrt(2))+(1/4)*EllipticPi(1/2-1/2*I,
(1/2)*sqrt(2))-(1/4*I)*EllipticPi(1/2+1/2*I,
(1/2)*sqrt(2))-(1/4)*EllipticPi(1/2+1/2*I,
(1/2)*sqrt(2))-(1/2*I)*EllipticF(sqrt(2),
(1/2)*sqrt(2))+(1/4*I)*EllipticPi(sqrt(2), 1/2-1/2*I,
(1/2)*sqrt(2))-(1/4)*EllipticPi(sqrt(2), 1/2-1/2*I,
(1/2)*sqrt(2))+(1/4*I)*EllipticPi(sqrt(2), 1/2+1/2*I,
(1/2)*sqrt(2))+(1/4)*EllipticPi(sqrt(2), 1/2+1/2*I,
(1/2)*sqrt(2)).
```

as the answer. WolframAlpha, however, computes

$$\int_0^{\pi/2} \sqrt{\sin^3(x)\cos(x)}\,\mathrm{d}x = \frac{\pi}{4\sqrt{2}} \tag{23}$$

$$\begin{aligned}&\approx 0.555360367269795780876\ldots\\ &985123757586712326827711\ldots\\ &117196127788567445086955\ldots\\ &543491374342388219158666\ldots\\ &84559565467\ldots,\end{aligned}$$

which is exactly our answer obtained analytically in (22). MATLAB confirms the numerical calculation made by WolframAlpha; it computes

$$\frac{\pi}{4\sqrt{2}} \approx 0.555360367269796. \tag{24}$$

Therefore, we have

$$\begin{aligned}\int_0^{\pi/2} \sqrt{\sin^3(x)\cos(x)}\,\mathrm{d}x &= \frac{1}{2}B\left(\frac{5}{4},\frac{3}{4}\right) = \frac{\Gamma(5/4)\,\Gamma(3/4)}{2\Gamma(2)}\\ &= \frac{\pi}{4\sqrt{2}} \approx 0.555360367269796.\end{aligned} \tag{25}$$

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4.

$$\int_0^1 x^2\left(1-x^2\right)^{3/2}\mathrm{d}x$$

In addition to its forms in (17) and (18), the B function can be written as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x. \tag{26}$$

To compute the integral in question, first let $u = x^2$. Then $\mathrm{d}u = 2x\mathrm{d}x$ and $x = u^{1/2}$. So, we get

$$\int_0^1 x^2\left(1-x^2\right)^{3/2}\mathrm{d}x = \frac{1}{2}\int_0^1 x(1-u)^{3/2}\,\mathrm{d}u = \frac{1}{2}\int_0^1 u^{1/2}(1-u)^{3/2}\,\mathrm{d}u. \tag{27}$$

Then, letting $p = 3/2$ and $q = 5/2$ in (26), we get

$$\int_0^1 x^2\left(1-x^2\right)^{3/2}\,dx = \frac{1}{2}B\left(\frac{3}{2},\frac{5}{2}\right). \tag{28}$$

So, from (18) it follows that

$$\begin{aligned}\int_0^1 x^2\left(1-x^2\right)^{3/2}\,dx &= \frac{1}{2}\frac{\Gamma(3/2)\,\Gamma(5/2)}{\Gamma(3/2+5/2)} = \frac{\Gamma(3/2)\,\Gamma(5/2)}{2\Gamma(4)}\\ &= \frac{\Gamma(3/2)\,\Gamma(5/2)}{2\cdot 3!} = \frac{1}{12}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)\\ &= \frac{1}{12}\Gamma\left(\frac{1}{2}+1\right)\Gamma\left(\frac{3}{2}+1\right)\\ &= \frac{1}{12}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)\\ &= \frac{1}{12}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\\ &= \frac{1}{32}\left[\Gamma\left(\frac{1}{2}\right)\right]^2.\end{aligned} \tag{29}$$

There is another Γ function identity:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{30}$$

From (29) and (30) we get

$$\int_0^1 x^2\left(1-x^2\right)^{3/2}\,dx = \frac{\pi}{32}. \tag{31}$$

`Maple` computes this result exactly, confirming our answer. MATLAB approximates $\pi/32$ as 0.098174770424681. Thus, we have

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$$\int_0^1 x^2\left(1-x^2\right)^{3/2}\,dx = \frac{\pi}{32} \approx 0.098174770424681. \tag{32}$$

9.

Prove $B(n,n) = B\left(n,\frac{1}{2}\right)/2^{2n-1}$. *Hint:* In (6.4), use the identity $2\sin\theta\cos\theta = \sin 2\theta$ and put $2\theta = \phi$. Use this result and (5.3) to derive the *duplication formula* for Γ functions:

$$\Gamma(2n) = \frac{1}{\sqrt{\pi}}2^{2n-1}\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right).$$

Check this formula for the case $n = \frac{1}{4}$ by using (5.4).

From (17) we know that

$$B(n,n) = 2\int_0^{\pi/2}(\sin\theta\cos\theta)^{2n-1}\,d\theta, \tag{33}$$

and from the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$ it follows that

$$B(n,n) = 2\int_0^{\pi/2}\left(\frac{1}{2}\sin 2\theta\right)^{2n-1}\mathrm{d}\theta = \frac{1}{2^{2n-2}}\int_0^{\pi/2}(\sin 2\theta)^{2n-1}\,\mathrm{d}\theta. \tag{34}$$

Let $\phi = 2\theta$. Then $\mathrm{d}\phi = 2\mathrm{d}\theta$. So, we have

$$\begin{aligned} B(n,n) &= \frac{1}{2^{2n-2}}\frac{1}{2}\int_0^{\pi}(\sin\phi)^{2n-1}\,\mathrm{d}\phi \\ &= \frac{1}{2^{2n-1}}\int_0^{\pi}(\sin\phi)^{2n-1}\,\mathrm{d}\phi \\ &= \frac{1}{2^{2n-1}}2\int_0^{\pi/2}(\sin\phi)^{2n-1}\,\mathrm{d}\phi \\ &= \frac{1}{2^{2n-1}}\left[2\int_0^{\pi/2}(\sin\phi)^{2n-1}(\cos\phi)^0\,\mathrm{d}\phi\right] \\ &= \frac{1}{2^{2n-1}}B\left(n,\frac{1}{2}\right). \quad \blacksquare \end{aligned} \tag{35}$$

Now, we know from (18) and (30) that

$$B(n,n) = \frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} \quad \text{and} \quad B(n,1/2) = \frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)} = \frac{\sqrt{2}\Gamma(n)}{\Gamma(n+1/2)}. \tag{36}$$

Therefore, from (35) and (36) it follows that

$$\Gamma(2n) = \frac{\Gamma(n)\Gamma(n)}{B(n,n)} = 2^{2n-1}\frac{\Gamma(n)\Gamma(n)}{B(n,1/2)} = \frac{1}{\sqrt{\pi}}2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right). \quad \blacksquare \tag{37}$$

Let $n = \frac{1}{4}$. Then by (30) we have

$$\Gamma(2n) = \Gamma\left(2\cdot\frac{1}{4}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{2}. \tag{38}$$

Also, by (21), we have

$$\begin{aligned} \frac{1}{\sqrt{\pi}}2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) &= \frac{1}{\sqrt{\pi}}2^{-1/2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \\ &= \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) \\ &= \frac{1}{\sqrt{2\pi}}\frac{\pi}{\sin\pi/4} \\ &= \frac{1}{\sqrt{2}}\frac{\sqrt{\pi}}{\sqrt{2}/2} \\ &= \sqrt{\pi}. \end{aligned} \tag{39}$$

Thus, we have verified (37) for the case $n = \frac{1}{4}$.

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3.
The figure is part of a cycloid with parametric equations

$$x = a(\theta + \sin\theta), \qquad y = a(1 - \cos\theta).$$

(The graph shown is like Figure 4.4 of Chapter 9 with the origin shifted to P_2.) Show that the time for a particle to slide without friction along the curve from (x_1, y_1) is given by

$$t = \sqrt{\frac{a}{g}} \int_0^{y_1} \frac{1}{\sqrt{y(y_1 - y)}} \mathrm{d}y.$$

Hint: Show that the arc length element is $\mathrm{d}s = \sqrt{2a/y}\,\mathrm{d}y$. Evaluate the integral to show that the time is independent of the starting height y_1.

We assume that the particle has initial conditions $y(0) = y_1$ and $v(0) = 0$. For the kinetic energy of the particle, we have

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2, \tag{40}$$

where s is the arc length of the curve parametrized by x and y. For the gravitational potential energy (after defining $h = 0$ at $y = y_1$) we have

$$V = mgh = mg(y - y_1). \tag{41}$$

Since $y(0) = y_1$, $V(0) = 0$, and since $v(0) = 0$, $T(0) = 0$. So, $E(0) = T(0) + V(0) = 0$. Since there is no friction, energy remains constant; the sum of T and V is always zero. That is,

$$T + V = \frac{1}{2}m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 + mg(y - y_1) = 0. \tag{42}$$

Therefore, we solve for $\mathrm{d}s/\mathrm{d}t$ to obtain

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{2g(y_1 - y)} \implies t = \int_0^t \mathrm{d}\tau = \int_{y=0}^{y=y_1} \frac{1}{\sqrt{2g(y_1 - y)}} \mathrm{d}s. \tag{43}$$

By definition, the arc length of the curve is given by

$$\begin{aligned} \mathrm{d}s &= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\ &= \sqrt{(a + a\cos\theta)^2 + (a\sin\theta)^2}\frac{1}{a\sin\theta}\mathrm{d}y \\ &= \sqrt{\frac{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}{\sin^2\theta}}\mathrm{d}y \\ &= \sqrt{\frac{2(1 + \cos\theta)}{1 - \cos^2\theta}}\mathrm{d}y = \sqrt{\frac{2}{1 - \cos\theta}}\mathrm{d}y = \sqrt{\frac{2a}{y}}\mathrm{d}y \end{aligned} \tag{44}$$

So, we have

$$t=\int_0^{y_1}\frac{1}{\sqrt{2g\left(y_1-y\right)}}\sqrt{\frac{2a}{y}}\mathrm{d}y=\sqrt{\frac{a}{g}}\int_0^{y_1}\frac{1}{\sqrt{y\left(y_1-y\right)}}\mathrm{d}y. \tag{45}$$

Let $y=y_1\sin^2\theta$. Then $\mathrm{d}y=2y_1\sin\theta\cos\theta\mathrm{d}\theta$. So, we have

$$t=\sqrt{\frac{a}{g}}\int_0^{\pi/2}\frac{2y_1\sin\theta\cos\theta}{y_1\sin\theta\cos\theta}\mathrm{d}\theta=2\sqrt{\frac{a}{g}}\int_0^{\pi/2}\mathrm{d}\theta=\sqrt{\frac{a}{g}}\pi. \tag{46}$$

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See Ricardo addendum

PROBLEMS, SECTION 11.12

1.
Expand the integrands of K and E [see (12.3)] in power series in $k^2\sin^2\theta$ (assuming small k), and integrate term by term to find power series approximations for the complete elliptic integrals K and E.

The K function (Complete Elliptic Integral of the first kind) is defined by

$$K\left(k\right)=\int_0^{\pi/2}\frac{1}{\sqrt{1-k^2\sin^2\theta}}\mathrm{d}\theta. \tag{47}$$

Suppose $|k|\ll 1$. Then $\left|k^2\sin^2\theta\right|\ll 1$. Let $\varepsilon=k^2\sin^2\theta$ and define a function f by $f\left(x\right)=x^{-1/2}$. Then

$$K\left(k\right)=\int_0^{\pi/2}f\left(1-\varepsilon\right)\mathrm{d}\theta. \tag{48}$$

We know that the Taylor approximation of f around x for a small ε is given by

$$f\left(x+\varepsilon\right)=\sum_{n\in\mathbb{N}}\frac{\varepsilon^n}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\left(x\right) \tag{49}$$

For our $f\left(x\right)=x^{-1/2}$, the subsequent derivatives are

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x}&=-\frac{1}{2}x^{-3/2}=-\frac{1}{2^1}x^{-(2(1)+1)/2}\\
\frac{\mathrm{d}^2f}{\mathrm{d}x^2}&=\frac{3}{4}x^{-5/2}=\frac{1\cdot 3}{2^2}x^{-(2(2)+1)/2}\\
\frac{\mathrm{d}^3f}{\mathrm{d}x^3}&=-\frac{15}{8}x^{-7/2}=-\frac{1\cdot 3\cdot 5}{2^3}x^{-(2(3)+1)/2}\\
\frac{\mathrm{d}^4f}{\mathrm{d}x^4}&=\frac{105}{16}x^{-9/2}=\frac{1\cdot 3\cdot 5\cdot 7}{2^4}x^{-(2(4)+1)/2}\\
&\vdots\\
\frac{\mathrm{d}^nf}{\mathrm{d}x^n}&=\left(-1\right)^n\left[\frac{1}{2^n}\prod_{i=1}^{n}\left(2i-1\right)\right]x^{-(2n+1)/2}.
\end{aligned} \tag{50}$$

Therefore,

$$f\left(x+\varepsilon\right)=\sum_{n\in\mathbb{N}}\left\{\frac{\left(-1\right)^n\varepsilon^n}{2^n n!}\left[\prod_{i=1}^{n}\left(2i-1\right)\right]x^{-(2n+1)/2}\right\}. \tag{51}$$

We are interested in the expansion of $f(1-\varepsilon)$, so we compute:

$$f(1-\varepsilon) = \sum_{n\in\mathbb{N}} \left[\frac{(-1)^n (-\varepsilon)^n}{2^n n!} \prod_{i=1}^{n} (2i-1) \right] \tag{52}$$
$$= \sum_{n\in\mathbb{N}} \left[\frac{\varepsilon^n}{2^n n!} \prod_{i=1}^{n} (2i-1) \right].$$

So, substituting (52) into (48) (and using equations (3), (17), (18), and (30)), we get

$$K(k) = \int_0^{\pi/2} \sum_{n\in\mathbb{N}} \left[\frac{\varepsilon^n}{2^n n!} \prod_{i=1}^{n} (2i-1) \right] \mathrm{d}\theta \tag{53}$$
$$= \sum_{n\in\mathbb{N}} \int_0^{\pi/2} \left[\frac{\varepsilon^n}{2^n n!} \prod_{i=1}^{n} (2i-1) \right] \mathrm{d}\theta$$
$$= \sum_{n\in\mathbb{N}} \left[\frac{1}{2^n n!} \prod_{i=1}^{n} (2i-1) \right] \int_0^{\pi/2} \varepsilon^n \mathrm{d}\theta$$
$$= \sum_{n\in\mathbb{N}} \left[\frac{k^{2n}}{2^n n!} \prod_{i=1}^{n} (2i-1) \right] \int_0^{\pi/2} (\sin\theta)^{2n}\, d\theta$$
$$= \sum_{n\in\mathbb{N}} \frac{k^{2n}}{2^{n+1} n!} B\left(\frac{2n+1}{2}, \frac{1}{2} \right) \prod_{i=1}^{n} (2i-1).$$
$$= \sum_{n\in\mathbb{N}} \frac{k^{2n}}{2^{n+1} n!} \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)} \prod_{i=1}^{n} (2i-1).$$
$$= \sum_{n\in\mathbb{N}} \frac{k^{2n}\sqrt{\pi}}{2^{n+1} (n!)^2} \Gamma\left(n+\frac{1}{2}\right) \prod_{i=1}^{n} (2i-1).$$

For $n \in \mathbb{N}$, let us examine $\Gamma(n+1/2)$:

$$\Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \tag{54}$$
$$\Gamma\left(2+\frac{1}{2}\right) = \Gamma\left(\frac{5}{2}\right) = \Gamma\left(1+\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3\sqrt{\pi}}{4}$$
$$\Gamma\left(3+\frac{1}{2}\right) = \Gamma\left(\frac{7}{2}\right) = \Gamma\left(1+\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{15\sqrt{\pi}}{8}$$
$$\Gamma\left(4+\frac{1}{2}\right) = \Gamma\left(\frac{9}{2}\right) = \Gamma\left(1+\frac{7}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{105\sqrt{\pi}}{16}$$
$$\vdots$$
$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} \prod_{i=1}^{n} (2i-1).$$

Plugging (54) into (53) and simplifying gives

$$K(k)=\frac{\pi}{2}\sum_{n\in\mathbb{N}}\frac{k^{2n}}{2^{2n}(n!)^2}\prod_{i=1}^{n}(2i-1)^2. \tag{55}$$

Now we look at the E function (Complete Elliptic Integral of the second kind). It is defined by

$$E(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta. \tag{56}$$

Suppose again that $|k|\ll 1$, so $\left|k^2\sin^2\theta\right|\ll 1$. Let $\varepsilon=k^2\sin^2\theta$ again as well and define a function g by $g(x)=x^{1/2}$. Then

$$E(k)=\int_0^{\pi/2} g(1-\varepsilon)\,\mathrm{d}\theta. \tag{57}$$

The derivatives of g are

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}x}&=\frac{1}{2}x^{-1/2}=\frac{1}{2^1}x^{-(2(1)-1)/2}\\ \frac{\mathrm{d}^2g}{\mathrm{d}x^2}&=-\frac{1}{4}x^{-3/2}=-\frac{1}{2^2}x^{-(2(2)-1)/2}\\ \frac{\mathrm{d}^3g}{\mathrm{d}x^3}&=\frac{3}{8}x^{-5/2}=\frac{1\cdot 3}{2^3}x^{-(2(3)-1)/2}\\ \frac{\mathrm{d}^4g}{\mathrm{d}x^4}&=-\frac{15}{16}x^{-7/2}=-\frac{1\cdot 3\cdot 5}{2^4}x^{-(2(4)-1)/2}\\ &\vdots\\ \frac{\mathrm{d}^ng}{\mathrm{d}x^n}&=(-1)^{n+1}\left[\frac{1}{2^n}\prod_{i=1}^{n-1}(2i-1)\right]x^{-(2n-1)/2}.\end{aligned} \tag{58}$$

So,

$$g(x+\varepsilon)=\sum_{n\in\mathbb{N}}\left\{\frac{(-1)^{n+1}\varepsilon^n}{2^n n!}\left[\prod_{i=1}^{n-1}(2i-1)\right]x^{-(2n-1)/2}\right\}. \tag{59}$$

We are only interested in the expansion of $g(1-\varepsilon)$, so we compute:

$$\begin{aligned}g(1-\varepsilon)&=\sum_{n\in\mathbb{N}}\left[\frac{(-1)^{n+1}(-\varepsilon)^n}{2^n n!}\prod_{i=1}^{n-1}(2i-1)\right]\\ &=\sum_{n\in\mathbb{N}}\left[-\frac{\varepsilon^n}{2^n n!}\prod_{i=1}^{n-1}(2i-1)\right].\end{aligned} \tag{60}$$

Now we substitute (60) into (57). We get

$$E(k) = \int_0^{\pi/2} \sum_{n\in\mathbb{N}} \left[-\frac{\varepsilon^n}{2^n n!} \prod_{i=1}^{n-1} (2i-1) \right] \mathrm{d}\theta \tag{61}$$

$$= \sum_{n\in\mathbb{N}} \left[-\frac{1}{2^n n!} \prod_{i=1}^{n-1} (2i-1) \right] \int_0^{\pi/2} \varepsilon^n \mathrm{d}\theta$$

$$= \sum_{n\in\mathbb{N}} \left[-\frac{k^{2n}}{2^n n!} \prod_{i=1}^{n-1} (2i-1) \right] \int_0^{\pi/2} (\sin\theta)^{2n}\, d\theta$$

$$= -\sum_{n\in\mathbb{N}} \frac{k^{2n}}{2^{n+1} n!} B\left(\frac{2n+1}{2}, \frac{1}{2}\right) \prod_{i=1}^{n-1} (2i-1).$$

$$= -\sum_{n\in\mathbb{N}} \frac{k^{2n}}{2^{n+1} n!} \frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)} \prod_{i=1}^{n-1} (2i-1).$$

$$= -\sum_{n\in\mathbb{N}} \frac{k^{2n}\sqrt{\pi}}{2^{n+1}(n!)^2} \Gamma\left(n+\frac{1}{2}\right) \prod_{i=1}^{n-1} (2i-1).$$

$$= -\frac{\pi}{2} \sum_{n\in\mathbb{N}} \frac{k^{2n}(2n-1)}{2^{2n}(n!)^2} \prod_{i=1}^{n-1} (2i-1)^2.$$

So, overall we have

$$K(k) = \frac{\pi}{2} \sum_{n\in\mathbb{N}} \frac{k^{2n}}{2^{2n}(n!)^2} \prod_{i=1}^{n} (2i-1)^2, = \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \ldots\right) \tag{62}$$

and

$$E(k) = -\frac{\pi}{2} \sum_{n\in\mathbb{N}} \frac{k^{2n}(2n-1)}{2^{2n}(n!)^2} \prod_{i=1}^{n-1} (2i-1)^2. = \frac{\pi}{2}\left(1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1}{2\cdot 4}\right)^2 3k^4 - \left(\frac{1\cdot 3}{2\cdot 4\cdot 6}\right)^2 \cdot 5k^6 - \ldots\right) \tag{63}$$

8 Adequate to show a few terms of taylor's series

17.
Write the integral in equation (12.7) as an elliptic integral and show that (12.8) gives its value. *Hints:* Write $\cos\theta = 1 - 2\sin^2\theta/2$ and a similar equation for $\cos\alpha$. Then make the change of variable $x = \sin(\theta/2)/\sin(\alpha/2)$.

Equation (12.7) is

$$\int_0^{\alpha} \frac{1}{\sqrt{\cos\theta - \cos\alpha}} \mathrm{d}\theta = \sqrt{\frac{2g}{l}} \frac{T_\alpha}{4}. \tag{64}$$

By the identity $\cos\theta = 1 - 2\sin^2\theta/2$, we get

$$\int_0^\alpha \frac{1}{\sqrt{\cos\theta - \cos\alpha}} \mathrm{d}\theta = \int_0^\alpha \frac{1}{\sqrt{1 - 2\sin^2\frac{\theta}{2} - 1 + 2\sin^2\frac{\alpha}{2}}} \mathrm{d}\theta \tag{65}$$

$$= \int_0^\alpha \frac{1}{\sqrt{2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)}} \mathrm{d}\theta$$

$$= \frac{\sqrt{2}}{2} \int_0^\alpha \frac{1}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} \mathrm{d}\theta$$

Let $x = \sin(\theta/2)/\sin(\alpha/2)$. Then $\mathrm{d}x = \frac{1}{2}\cos(\theta/2)/\sin(\alpha/2)\,\mathrm{d}\theta$ and $\sin^2(\theta/2) = x^2\sin^2(\alpha/2)$. So we have

$$\int_0^\alpha \frac{1}{\sqrt{\cos\theta - \cos\alpha}} \mathrm{d}\theta = \frac{\sqrt{2}}{2} \int_0^\alpha \frac{1}{\sqrt{\sin^2\frac{\alpha}{2} - x^2\sin^2\frac{\alpha}{2}}} \mathrm{d}\theta \tag{66}$$

$$= \sqrt{2} \int_0^1 \frac{\sin(\alpha/2)}{\cos(\theta/2)\sqrt{(1 - x^2)\sin^2\frac{\alpha}{2}}} \mathrm{d}x$$

$$= \sqrt{2} \int_0^1 \frac{1}{\cos(\theta/2)\sqrt{(1 - x^2)}} \mathrm{d}x.$$

Now we use the identity $\cos(\theta/2) = \sqrt{1 - \sin^2(\theta/2)}$:

$$\int_0^\alpha \frac{1}{\sqrt{\cos\theta - \cos\alpha}} \mathrm{d}\theta = \sqrt{2} \int_0^1 \frac{1}{\sqrt{1 - \sin^2(\theta/2)}\sqrt{(1 - x^2)}} \mathrm{d}x \tag{67}$$

$$= \sqrt{2} \int_0^1 \frac{1}{\sqrt{1 - x^2\sin^2(\alpha/2)}\sqrt{(1 - x^2)}} \mathrm{d}x.$$

An alternate way of writing the K function is

$$K(k) = \int_0^1 \frac{1}{\sqrt{1 - t^2}\sqrt{1 - k^2 t^2}} \mathrm{d}t. \tag{68}$$

We see that (67) is exactly (68) when $k = \sin(\alpha/2)$. So,

$$\int_0^\alpha \frac{1}{\sqrt{\cos\theta - \cos\alpha}} \mathrm{d}\theta = \sqrt{2} K\left(\sin\left(\frac{\alpha}{2}\right)\right). \tag{69}$$

5

HW #2

11.5 1. a. $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \sqrt{\pi}/2$

b. $\Gamma(-\frac{1}{2}) = \frac{1}{(-1/2)}\Gamma(1/2) = -2\sqrt{\pi}$

3 c. $\Gamma(-3/2) = \frac{1}{(-3/2)}\Gamma(-\frac{1}{2}) = \frac{4}{3}\sqrt{\pi}$

11.11.3 Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$

$\ln(n!) \sim \ln(n^n e^{-n}\sqrt{2\pi n}) = n\ln(n) - n + \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(n)$

For $n = 10^{23}$, $n\ln(n) \approx 5.296\times 10^{24}$

$\frac{1}{2}\ln(2\pi) \simeq 0.9189$ $\qquad \frac{1}{2}\ln(n) \approx 26.48$

3 Thus, the first two terms are most significant and

$\ln(n!) \sim n\ln(n) - n \qquad \ln(10^{23}) \sim 5.196\times 10^{24}$

Note: Maple fails to find $\ln(n!)$

11.12.9 $\displaystyle\int_{-1/2}^{1/2}\frac{dt}{\sqrt{1-t^2}\sqrt{4-3t^2}} = \frac{1}{2}\int_{-1/2}^{1/2}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\frac{3}{4}t^2}}$

Since elliptic integrals are even $\displaystyle\int_{-1/2}^{0}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\frac{3}{4}t^2}} = \int_{0}^{1/2}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\frac{3}{4}t^2}}$

4 $\displaystyle\therefore \frac{1}{2}\int_{-1/2}^{1/2}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\frac{3}{4}t^2}} = \int_{0}^{1/2}\frac{dt}{\sqrt{1-t^2}\sqrt{1-\frac{3}{4}t^2}} = F(\tfrac{1}{2}; \tfrac{\sqrt{3}}{2}) \approx 0.54223$