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Math 342B Assignment 4

Problems, Section 7.5

In each of the following problems you are given a function on the interval $-\pi < x < \pi$. Sketch several periods of the corresponding periodic function of period 2π. Expand the periodic function in a sine-cosine Fourier series.

2.

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \frac{\pi}{2}, \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$$

Figure 1.(a)[1] shows two periods of $f(x)$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi/2} \mathrm{d}x = \frac{\pi}{2\pi} = \frac{1}{2}. \tag{1}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi/2} \cos(nx)\mathrm{d}x \tag{2}$$

$$= \frac{1}{n\pi}[\sin(nx)]_0^{\pi/2} = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n \equiv 0 \pmod 2, \\ \dfrac{1}{n\pi}, & n \equiv 1 \pmod 4, \\ -\dfrac{1}{n\pi}, & n \equiv 3 \pmod 4. \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi/2} \sin(nx)\mathrm{d}x \tag{3}$$

$$= -\frac{1}{n\pi}[\cos(nx)]_0^{\pi/2} = \frac{1}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right] = \begin{cases} 0, & n \equiv 0 \pmod 4, \\ \dfrac{1}{n\pi}, & n \equiv 1 \pmod 2, \\ \dfrac{2}{n\pi}, & n \equiv 2 \pmod 4. \end{cases}$$

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w/ graph

[1] All figures are at the end.

From (1), (2), and (3) we get

$$(4)\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nx) + \frac{1}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right] \sin(nx) \right\}$$

$$= \frac{1}{4} + \frac{1}{\pi}\left(\cos(x) - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}\cos(7x) + \cdots\right)$$

$$+ \frac{1}{\pi}\left(\sin(x) + \sin(2x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{3}\sin(6x) + \cdots\right)$$

$$= \frac{1}{4} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{(-1)^{k+1}\cos((2k-1)x) + \sin((2k-1)x) + \sin((4k-2)x)}{2k-1}\right].$$

7.

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$$

FIGURE 2.(a) shows two periods of $f(x)$.

$$(5)\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x\mathrm{d}x = \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi} = \frac{\pi}{2}.$$

$$(6)\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\mathrm{d}x.$$

Let $u = x$ and $\mathrm{d}v = \cos(nx)\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{1}{n}\sin(nx)$. Using (6) and integration by parts for definite integrals ($\int_a^b u\mathrm{d}v = uv|_a^b - \int_a^b v\mathrm{d}u$), we get

$$(7)\quad a_n = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\mathrm{d}x = \frac{1}{\pi}\left(\left[\frac{1}{n}x\sin(nx)\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\mathrm{d}x\right)$$

$$= \frac{1}{n^2\pi}[\cos(nx)]_0^{\pi} = \frac{1}{n^2\pi}(\cos(n\pi) - 1) = \begin{cases} -\frac{2}{n^2\pi}, & n \equiv 1 \pmod 2, \\ 0, & n \equiv 0 \pmod 2. \end{cases}$$

$$(8)\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\mathrm{d}x.$$

Let $u = x$ and $\mathrm{d}v = \sin(nx)\mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$ and $v = -\frac{1}{n}\cos(nx)$. Using (8) and integration by parts we get

$$(9)\quad b_n = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\mathrm{d}x = \frac{1}{\pi}\left(\left[-\frac{1}{n}x\cos(nx)\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\mathrm{d}x\right)$$

$$= -\frac{1}{n\pi}\left(\pi\cos(n\pi) + \frac{1}{n}[\sin(nx)]_0^{\pi}\right) = -\frac{1}{n}\cos(n\pi) = \begin{cases} \frac{1}{n}, & n \equiv 1 \pmod 2, \\ -\frac{1}{n}, & n \equiv 0 \pmod 2. \end{cases}$$

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w/ graph

From (5), (7), and (9) we get

$$
\begin{aligned}
(10)\qquad f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{n^2\pi} (\cos(nx) - 1)\cos(nx) + \frac{1}{n} \cos(n\pi)\sin(nx) \right] \\
&= \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(x) + \frac{1}{9}\cos(3x) + \frac{1}{25}\cos(5x) + \cdots \right) \\
&\quad + \sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \cdots \\
&= \frac{\pi}{4} + \sum_{k=1}^{\infty} \left[\frac{(-1)^{k+1}}{k} \sin(kx) - \frac{2}{(2k-1)^2\pi} \cos((2k-1)x) \right].
\end{aligned}
$$

11.

$$
f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases}
$$

FIGURE 3.(a) shows two periods of $f(x)$.

$$
(11)\qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_0^{\pi} \sin(x)\mathrm{d}x = -\frac{1}{\pi}[\cos(x)]_0^{\pi} = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}.
$$

Let $I : \mathbb{R}^4 \to \mathbb{R}$ be defined by

$$
(12)\qquad I(m,n,a,b) = \int_a^b \sin(mx)\cos(nx)\mathrm{d}x
$$

with $n, m, a, b \in \mathbb{R}$ and $a < b$. Then,

$$
\begin{aligned}
(13)\qquad I(m,n,a,b) &= \int_a^b \frac{\mathrm{e}^{imx} - \mathrm{e}^{-imx}}{2i} \cdot \frac{\mathrm{e}^{inx} + \mathrm{e}^{-inx}}{2} \mathrm{d}x \\
&= \frac{1}{4i}\int_a^b \left[\mathrm{e}^{i(m+n)x} - \mathrm{e}^{-i(m+n)x} + \mathrm{e}^{i(m-n)x} - \mathrm{e}^{-i(m-n)x} \right] \mathrm{d}x \\
&= \frac{1}{2}\int_a^b [\sin((m+n)x) + \sin((m-n)x)]\mathrm{d}x \\
&= -\frac{1}{2}\left[\frac{1}{m+n}\cos((m+n)x) + \frac{1}{m-n}\cos((m-n)x) \right]_a^b \\
&= \frac{1}{2}\left[\frac{1}{m+n}\cos((m+n)a) - \frac{1}{m+n}\cos((m+n)b) \right. \\
&\quad \left. + \frac{1}{m-n}\cos((m-n)a) - \frac{1}{m-n}\cos((m-n)b) \right].
\end{aligned}
$$

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w/ graph

To solve for the Fourier coefficients of $f(x)$, we use (13) to evaluate the integrands inside the a_n terms:

$$
\begin{aligned}
(14)\qquad a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi}\sin(x)\cos(nx)\mathrm{d}x = \frac{1}{\pi}I(1,n,0,\pi)\\
&= \frac{1}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n+1}\cos((n+1)\pi) - \frac{1}{n-1} + \frac{1}{n-1}\cos((n+1)\pi)\right]\\
&= \frac{1}{2\pi}\left[-\frac{2}{n^2-1} + \frac{2}{n^2-1}\cos((n+1)\pi)\right]\\
&= \frac{1}{(n^2-1)\pi}(\cos((n+1)\pi)-1) = \begin{cases} -\dfrac{2}{(n^2-1)\pi}, & n\equiv 0 \pmod 2,\\ 0, & n \equiv 1 \pmod 2.\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
(15)\qquad b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi}\sin(x)\sin(nx)\mathrm{d}x\\
&= \frac{1}{\pi}\int_{0}^{\pi}\frac{\mathrm{e}^{ix}-\mathrm{e}^{-ix}}{2i}\cdot\frac{\mathrm{e}^{inx}-\mathrm{e}^{-inx}}{2i}\mathrm{d}x\\
&= -\frac{1}{4\pi}\int_{0}^{\pi}\left[\mathrm{e}^{i(n+1)x}+\mathrm{e}^{-i(n+1)x}-\mathrm{e}^{i(n-1)x}-\mathrm{e}^{-i(n-1)x}\right]\mathrm{d}x\\
&= \frac{1}{2\pi}\int_{0}^{\pi}[\cos((n-1)x)-\cos((n+1)x)]\mathrm{d}x\\
&= \frac{1}{2\pi}\left[\frac{1}{n-1}\sin((n-1)x) - \frac{1}{n+1}\sin((n+1)x)\right]_0^{\pi} = 0 \quad (n>1).
\end{aligned}
$$

$$
\begin{aligned}
(16)\qquad b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(x)\mathrm{d}x = \frac{1}{\pi}\int_{0}^{\pi}\sin^2(x)\mathrm{d}x\\
&= \frac{1}{2\pi}\int_{0}^{\pi}[1-\cos 2x]\mathrm{d}x = \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{2}.
\end{aligned}
$$

From (11), (14), (15), and (16) we get

$$
\begin{aligned}
(17)\qquad f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]\\
&= \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{3}\cos(2x) + \frac{1}{15}\cos(4x) + \frac{1}{35}\cos(6x) + \cdots\right) + \frac{1}{2}\sin(x)\\
&= \frac{1}{\pi} + \frac{1}{2}\sin(x) - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos(2kx).
\end{aligned}
$$

PROBLEMS, SECTION 7.8

You are given one period of a function. Sketch several periods of the function and expand it in a sine-cosine Fourier series, and in a complex exponential Fourier series.

11.a.

$$f(x) = x^2, \quad -\pi < x < \pi.$$

FIGURE 4.(a) shows two periods of $f(x)$. First, we find the sine-cosine Fourier series. It is clear from FIGURE 4.(a) that $f(x)$ is an even function—its reflection about the y-axis is a symmetry. Thus, we only need to find the a_0 and a_n terms.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\mathrm{d}x = \frac{1}{\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3}. \tag{18}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\mathrm{d}x. \tag{19}$$

Let $u = x^2$ and $\mathrm{d}v = \cos(nx)\mathrm{d}x$ so that $\mathrm{d}u = 2x\mathrm{d}x$ and $v = \frac{1}{n}\sin(nx)$. Thus, by integration by parts we have

$$a_n = \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx)\right]_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin(nx)\mathrm{d}x = -\frac{2}{n\pi}\int_{-\pi}^{\pi} x\sin(nx)\mathrm{d}x. \tag{20}$$

Let $u = x$ and $\mathrm{d}v = \sin(nx)\mathrm{d}x$ so that $\mathrm{d}u = \mathrm{d}x$ and $v = -\frac{1}{n}\cos(nx)$. Then we have

$$\begin{aligned} a_n &= \frac{2}{n\pi}\left[\frac{x}{n}\cos(nx)\right]_{-\pi}^{\pi} + \frac{2}{n^2\pi}\int_{-\pi}^{\pi}\cos(nx)\mathrm{d}x \\ &= \frac{2}{n^2\pi}(\pi\cos(n\pi) + \pi\cos(-n\pi)) = \frac{4(-1)^n}{n^2}. \end{aligned} \tag{21}$$

Thus, from (18) and (21) we have

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) \\ &= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx). \end{aligned} \tag{22}$$

Now we find the exponential Fourier series of $f(x)$:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-inx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\mathrm{e}^{-inx}\mathrm{d}x. \tag{23}$$

Let $u = x^2$ and $\mathrm{d}v = \mathrm{e}^{-inx}\mathrm{d}x$. Then $\mathrm{d}u = 2x\mathrm{d}x$ and $v = \frac{i}{n}\mathrm{e}^{-inx}$. So, we have

$$c_n = \frac{i}{2n\pi}\left[\left[x^2\mathrm{e}^{-inx}\right]_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi} x\mathrm{e}^{-inx}\mathrm{d}x\right]. \tag{24}$$

Let $u = x$, $\mathrm{d}v = \mathrm{e}^{-inx}\mathrm{d}x$ so that $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{i}{n}\mathrm{e}^{-inx}$. So,

$$(25)\qquad c_n = \frac{i}{2n\pi}\left[\left[x^2\mathrm{e}^{-inx}\right]_{-\pi}^{\pi} - \frac{2i}{n}\left(\left[x\mathrm{e}^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\mathrm{e}^{-inx}\mathrm{d}x\right)\right]$$
$$= \frac{i}{2n\pi}\left[\left[x^2\mathrm{e}^{-inx}\right]_{-\pi}^{\pi} - \frac{2i}{n}\left(\left[x\mathrm{e}^{-inx}\right]_{-\pi}^{\pi} - \frac{i}{n}\left[\mathrm{e}^{-inx}\right]_{-\pi}^{\pi}\right)\right]$$
$$= \frac{i}{2n\pi}\left[\pi^2\mathrm{e}^{-in\pi} - \pi^2\mathrm{e}^{in\pi} - \frac{2i}{n}\left(\pi\mathrm{e}^{-in\pi} + \pi\mathrm{e}^{in\pi} - \frac{i}{n}\left(\mathrm{e}^{-in\pi} - \mathrm{e}^{in\pi}\right)\right)\right]$$
$$= \frac{i}{2n\pi}\left[\pi^2\mathrm{e}^{-in\pi} - \pi^2\mathrm{e}^{in\pi} - \frac{2i\pi}{n}\mathrm{e}^{-in\pi} - \frac{2i\pi}{n}\mathrm{e}^{in\pi} - \frac{2}{n^2}\mathrm{e}^{-in\pi} + \frac{2}{n^2}\mathrm{e}^{in\pi}\right]$$
$$= \frac{i}{2n\pi}\left[-2i\pi^2\sin(n\pi) - \frac{4i\pi}{n}\cos(n\pi) + \frac{4i}{n^2}\sin(n\pi)\right]$$
$$= \frac{\pi}{n}\sin(n\pi) + \frac{2}{n^2}\cos(n\pi) - \frac{2}{n^3\pi}\sin(n\pi) = \frac{2(-1)^n}{n^2}, \quad n \neq 0.$$

$$(26)\qquad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\mathrm{d}x = \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_{-\pi}^{\pi} = \frac{\pi^2}{3}.$$

So, from (25) and (26) we have

$$(27)\qquad f(x) = \sum_{n=-\infty}^{\infty} c_n\mathrm{e}^{-inx} = \frac{\pi^3}{3} + 2\sum_{n\neq 0}\frac{(-1)^n}{n^2}\mathrm{e}^{-inx}.$$

Thus, we have the sine-cosine Fourier series expansion of $f(x)$:

$$(28)\qquad f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

and the exponential Fourier series expansion of $f(x)$:

$$(29)\qquad f(x) = \frac{\pi^3}{3} + 2\sum_{n\neq 0}\frac{(-1)^n}{n^2}\mathrm{e}^{-inx}.$$

The two forms are equivalent since they share the same constant term and the infinite sums in both are equal:

$$(30)\qquad 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx) = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{\mathrm{e}^{inx} + \mathrm{e}^{-inx}}{2}\right)$$
$$= 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\mathrm{e}^{inx} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\mathrm{e}^{-inx}$$
$$= 2\sum_{n=-\infty}^{-1}\frac{(-1)^n}{n^2}\mathrm{e}^{-inx} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\mathrm{e}^{-inx}$$
$$= 2\sum_{n\neq 0}\frac{(-1)^n}{n^2}\mathrm{e}^{-inx}.$$

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w/ graph

PROBLEMS, SECTION 7.9

Each function is given over one period. For each function, sketch several periods and decide whether it is even or odd. Then use (9.4) or (9.5) to expand it in an appropriate Fourier series.

6.

$$f(x) = \begin{cases} -1, & -l < x < 0, \\ 1, & 0 < x < l. \end{cases}$$

FIGURE 5.(a) shows two periods of $f(x)$. It is clear from FIGURE 5.(a) that $f(x)$ is an odd function—it has the property $f(-x) = -f(x)$. Thus, we only need to find the sine Fourier series expansion.

$$(31) \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}x\right)\mathrm{d}x = \frac{1}{l}\int_{0}^{l} \sin\left(\frac{n\pi}{l}x\right)\mathrm{d}x - \frac{1}{l}\int_{-l}^{0} \sin\left(\frac{n\pi}{l}x\right)\mathrm{d}x$$
$$= \frac{2}{l}\int_{0}^{l} \sin\left(\frac{n\pi}{l}x\right) = \frac{2}{n\pi}[1-\cos(n\pi)] = \begin{cases} \frac{4}{n\pi}, & n \equiv 1 \pmod 2, \\ 0 & n \equiv 0 \pmod 2. \end{cases}$$

So, from (31) we get

$$(32) \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$
$$= \frac{4}{\pi}\left[\sin\left(\frac{\pi}{l}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{l}x\right) + \frac{1}{5}\sin\left(\frac{5\pi}{l}x\right) + \cdots\right]$$
$$= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin\left(\frac{(2k-1)\pi}{l}x\right).$$

5 w/ graph

10.

$$f(x) = |x|, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

FIGURE 6.(a) shows two periods of $f(x)$, and it suggests that $f(x)$ is an even function—it is symmetric about the y-axis. Thus, we only need to find the cosine Fourier series expansion.

$$(33) \qquad a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\mathrm{d}x = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} |x|\mathrm{d}x = \frac{4}{\pi}\int_{0}^{\pi/2} x\mathrm{d}x = \frac{2}{\pi}\left[x^2\right]_0^{\pi/2} = \frac{\pi}{2}.$$

$$(34) \qquad a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos(2nx)\mathrm{d}x = \frac{4}{\pi}\int_{0}^{\pi/2} x\cos(2nx)\mathrm{d}x.$$

5 w/ graph

Let $u = x$ and $\mathrm{d}v = \cos(2nx)\mathrm{d}x$. Then, $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{1}{2n}\sin(2nx)$. Integrating by parts, we get

$$a_n = \frac{4}{\pi}\left(\frac{1}{2n}[x\sin(2nx)]_0^{\pi/2} - \frac{1}{2n}\int_0^{\pi/2}\sin(2nx)\mathrm{d}x\right) \tag{35}$$

$$= \frac{1}{n^2\pi}[\cos(2nx)]_0^{\pi/2} = \frac{1}{n^2\pi}[\cos(n\pi) - 1] = \begin{cases} -\dfrac{2}{n^2\pi}, & n \equiv 1 \pmod 2, \\ 0 & n \equiv 0 \pmod 2. \end{cases}$$

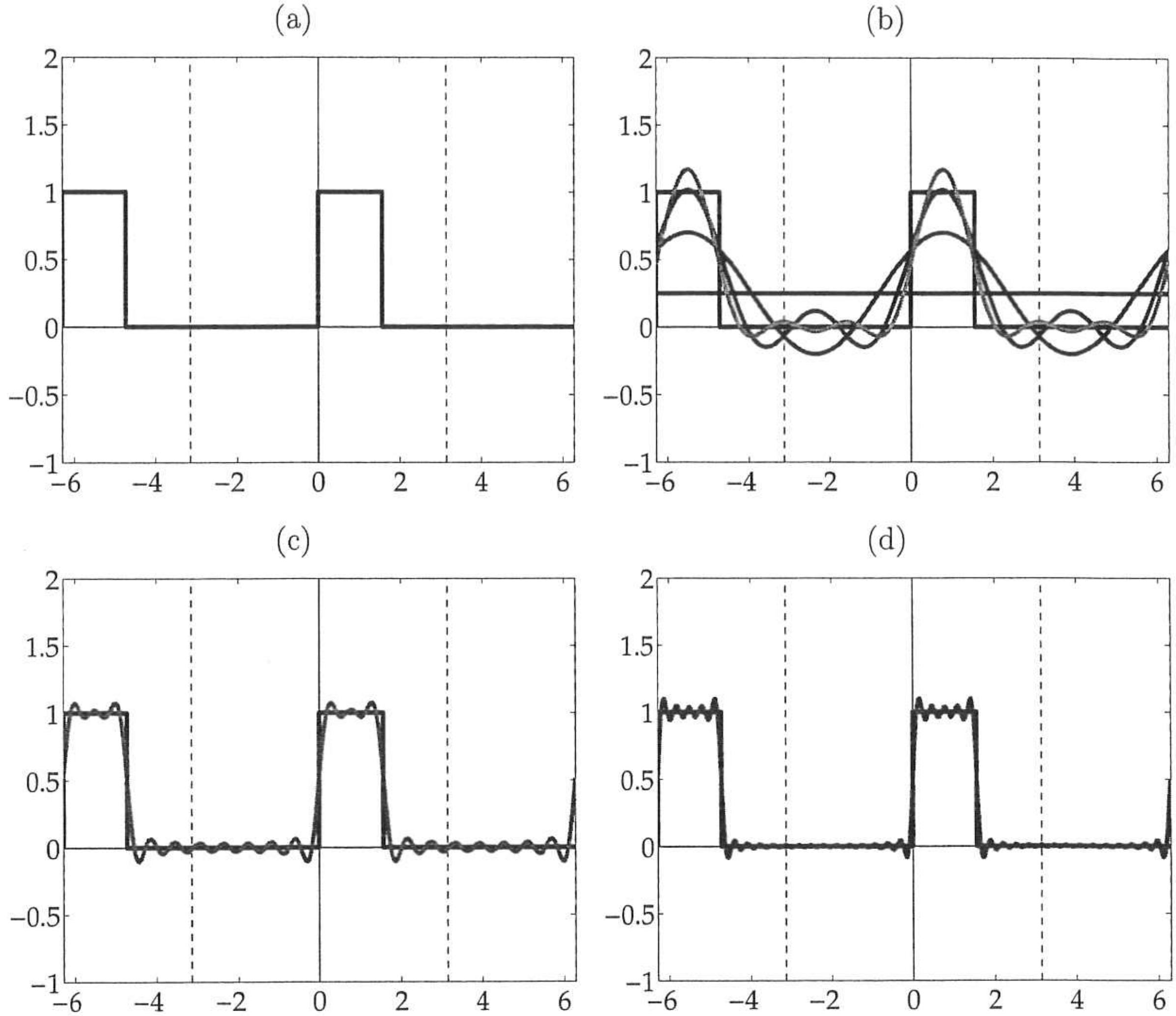
So, from (33) and (35) we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2nx) \tag{36}$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\cos(2x) + \frac{1}{9}\cos(6x) + \frac{1}{25}\cos(10x) + \cdots\right)$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos((4k-2)x).$$

Figures



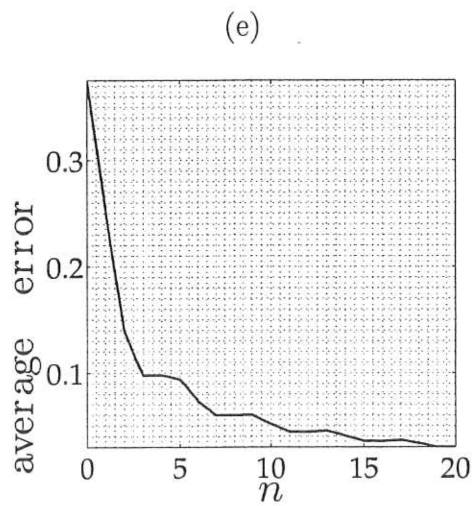
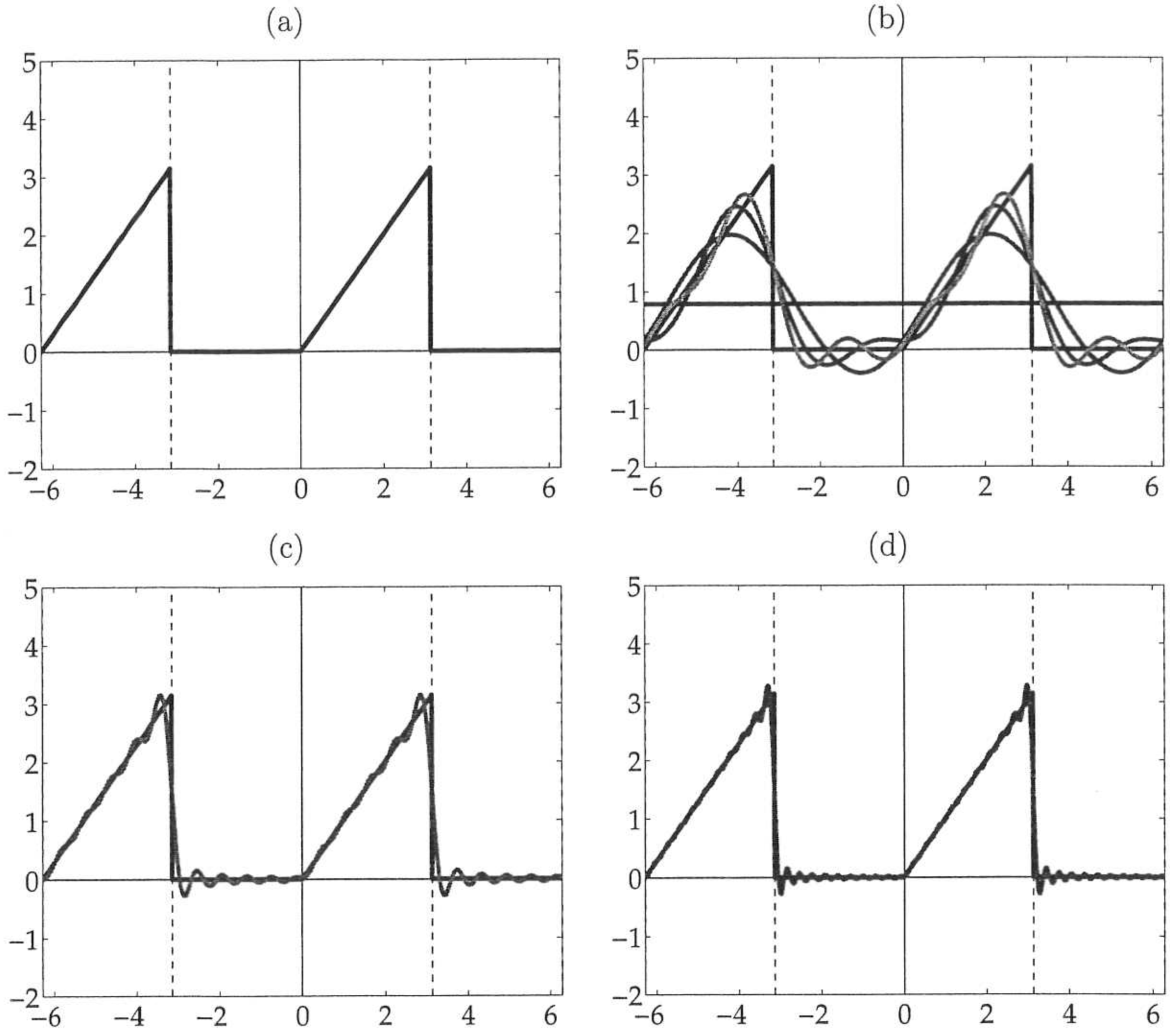


FIGURE 1.(a) Two periods of $f(x)$ from Problem 7.5.2. (b) Fourier expansion of $f(x)$ up to $n = 0$ (blue), $n = 1$ (green), $n = 2$ (red), and $n = 3$ (cyan). (c) $n = 10$. (d) $n = 20$. The distance between the dashed vertical lines is half of a period of $f(x)$. (e) Average of the sum of the differences of the real $f(x)$ and the Fourier series expansion from $n = 0$ to $n = 20$.



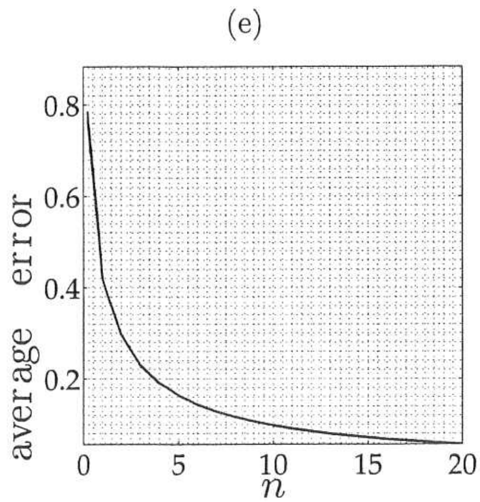
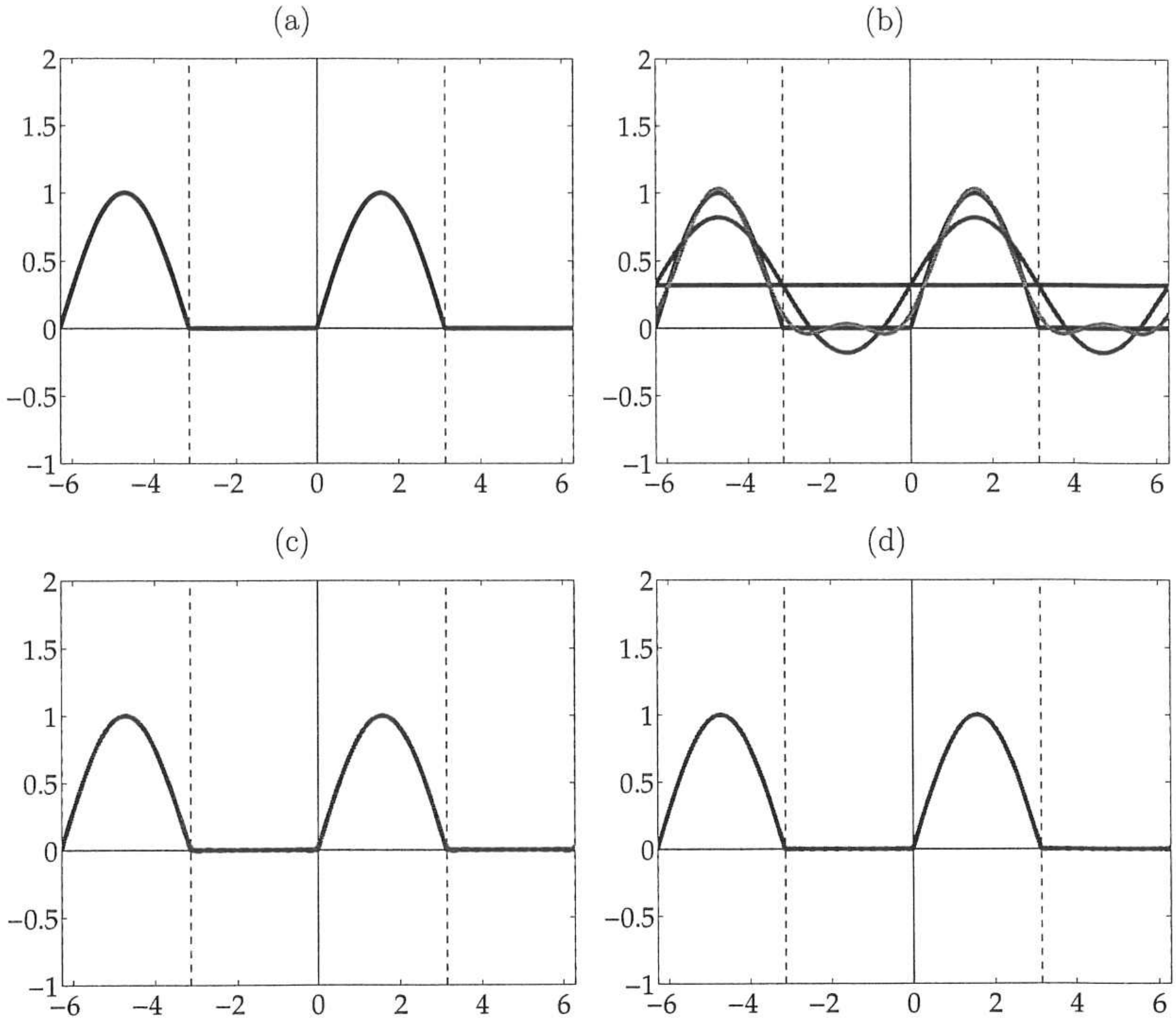


FIGURE 2.(a) Two periods of $f(x)$ from Problem 7.5.7. (b) Fourier expansion of $f(x)$ up to $n = 0$ (blue), $n = 1$ (green), $n = 2$ (red), and $n = 3$ (cyan). (c) $n = 10$. (d) $n = 20$. The distance between the dashed vertical lines is half of a period of $f(x)$. (e) Average of the sum of the differences of the real $f(x)$ and the Fourier series expansion from $n = 0$ to $n = 20$.



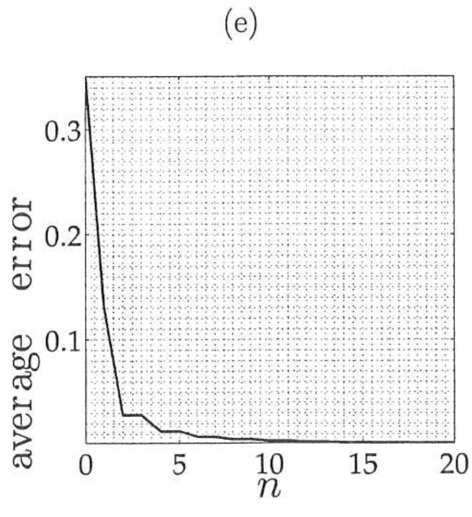
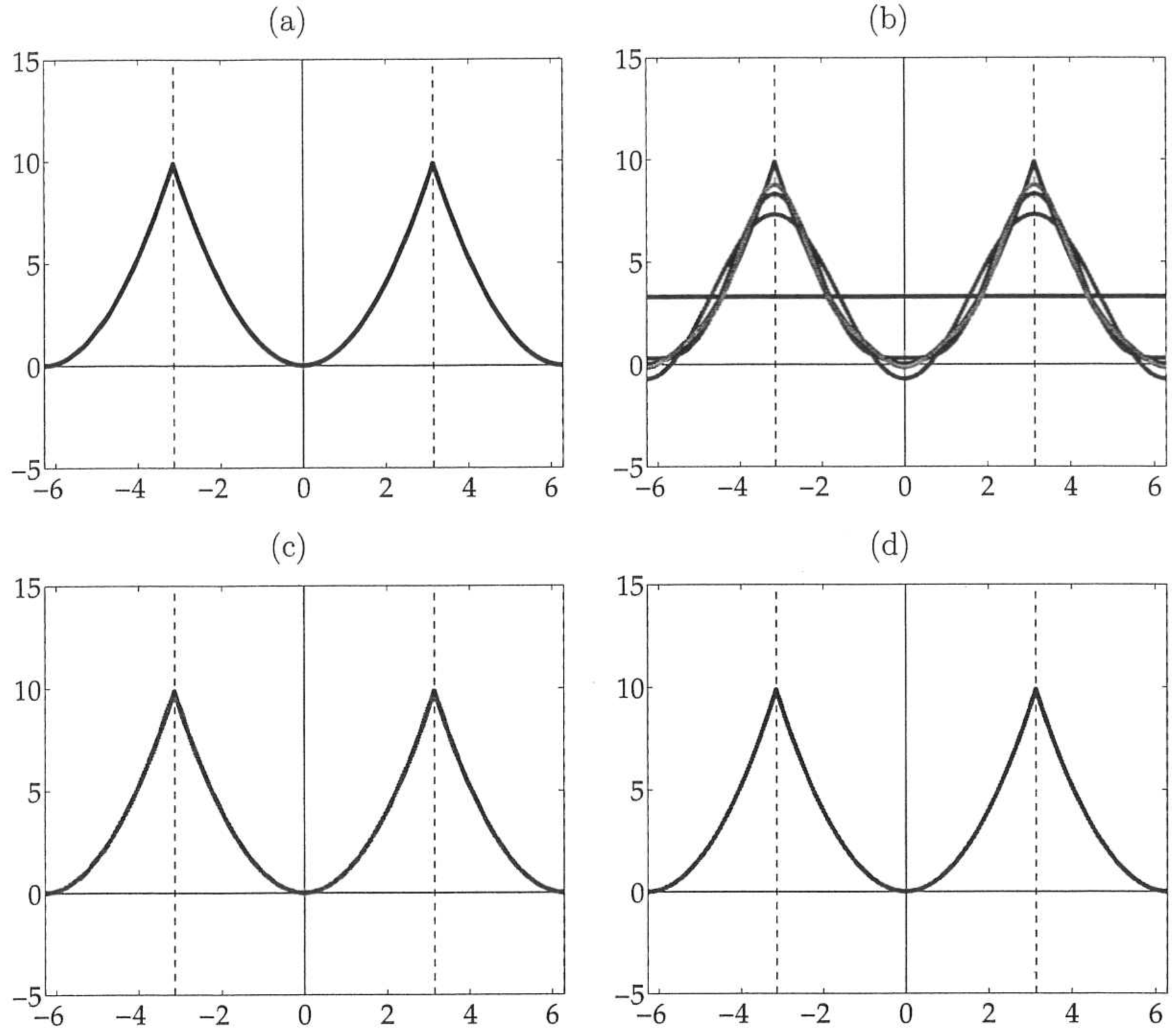


FIGURE 3.(a) Two periods of $f(x)$ from Problem 7.5.11. (b) Fourier expansion of $f(x)$ up to $n = 0$ (blue), $n = 1$ (green), $n = 2$ (red), and $n = 3$ (cyan). (c) $n = 10$. (d) $n = 20$. The distance between the dashed vertical lines is half of a period of $f(x)$. (e) Average of the sum of the differences of the real $f(x)$ and the Fourier series expansion from $n = 0$ to $n = 20$.



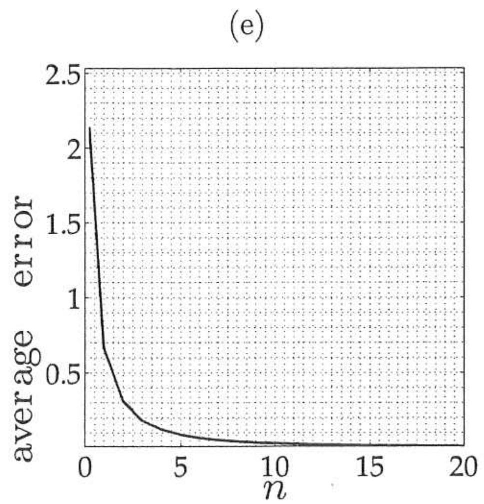
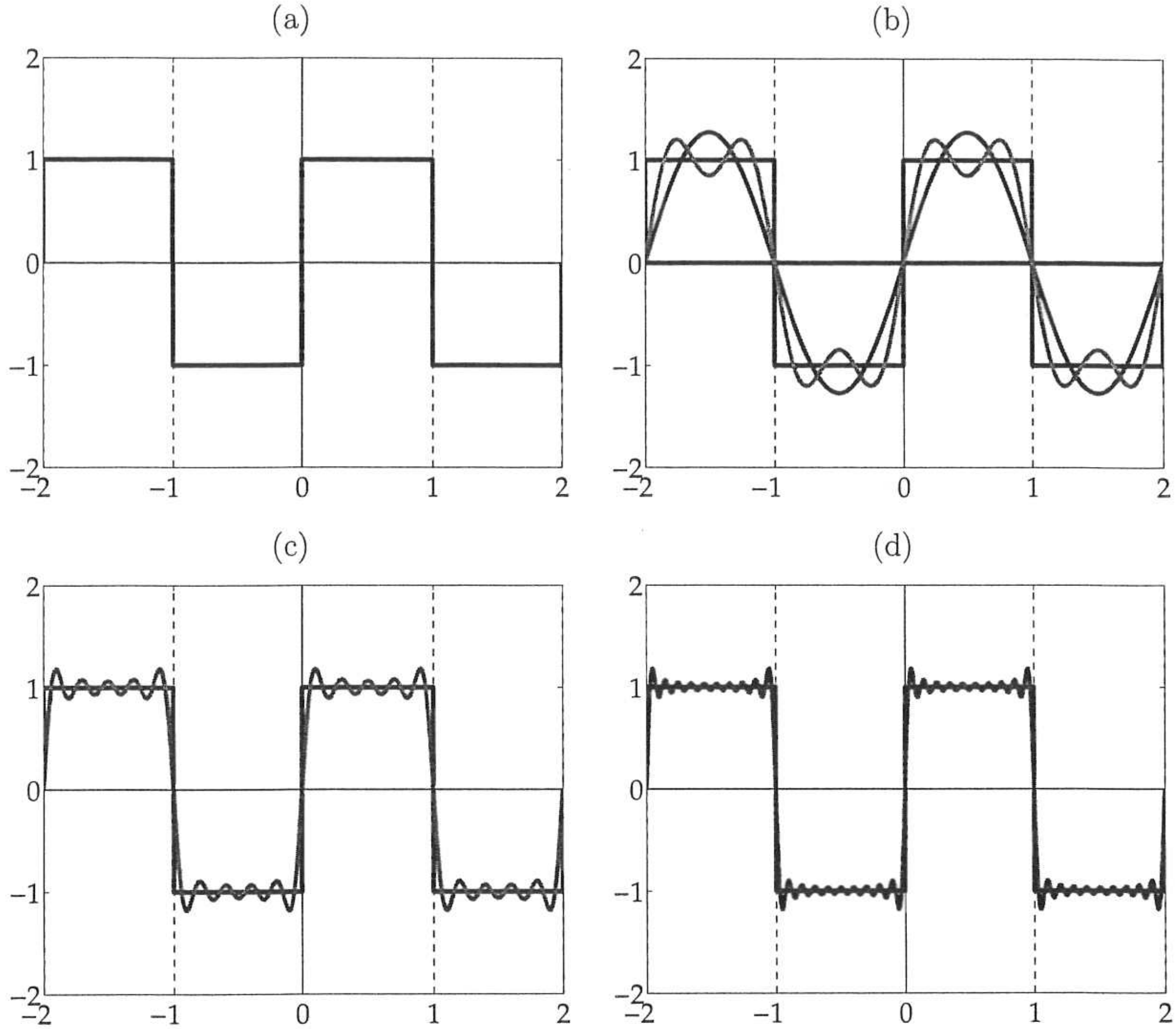


FIGURE 4.(a) Two periods of $f(x)$ from Problem 7.8.11.a. (b) Fourier expansion of $f(x)$ up to $n = 0$ (blue), $n = 1$ (green), $n = 2$ (red), and $n = 3$ (cyan). (c) $n = 10$. (d) $n = 20$. The distance between the dashed vertical lines is half of a period of $f(x)$. (e) Average of the sum of the differences of the real $f(x)$ and the Fourier series expansion from $n = 0$ to $n = 20$.



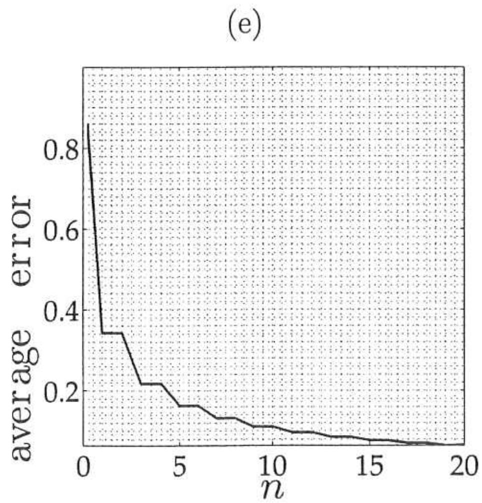


FIGURE 5.(a) Two periods of $f(x)$ from Problem 7.9.6. with $l = 1$. (b) Fourier expansion of $f(x)$ up to $n = 0$ (blue), $n = 1$ (green), $n = 2$ (red), and $n = 3$ (cyan). (c) $n = 10$. (d) $n = 20$. The distance between the dashed vertical lines is half of a period of $f(x)$. (e) Average of the sum of the differences of the real $f(x)$ and the Fourier series expansion from $n = 0$ to $n = 20$.

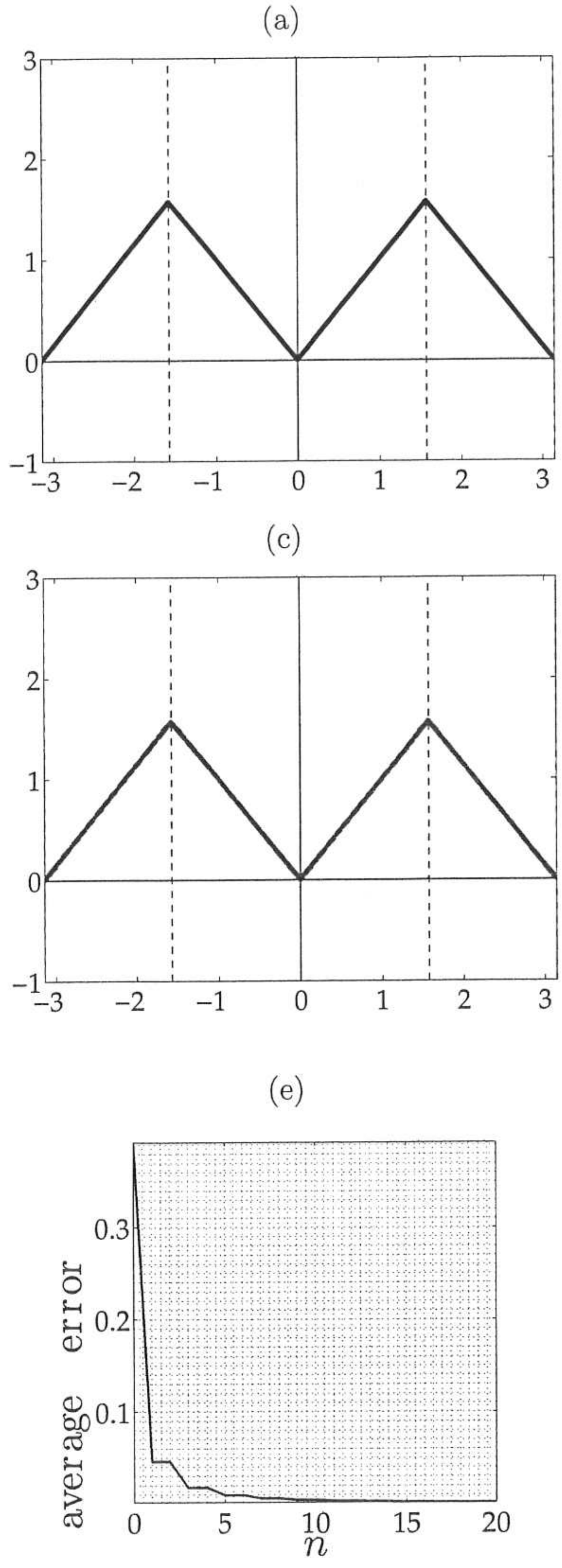


FIGURE 6.(a) Two periods of $f(x)$ from Problem 7.9.10. (b) Fourier expansion of $f(x)$ up to $n = 0$ (blue), $n = 1$ (green), $n = 2$ (red), and $n = 3$ (cyan). (c) $n = 10$. (d) $n = 20$. The distance between the dashed vertical lines is half of a period of $f(x)$. (e) Average of the sum of the differences of the real $f(x)$ and the Fourier series expansion from $n = 0$ to $n = 20$.