HW #8

$\boxed{14.2.12}$ $f(z) = \frac{z}{z^2+1} = \frac{(x+iy)\quad(x^2-y^2+1-2xyi)}{(x^2-y^2+1+2xyi)(x^2-y^2+1-2xyi)}$

$$= \frac{x^3+xy^2+x+i(y-x^2y-y^3)}{(x^2-y^2+1)^2+4x^2y^2}$$

$$\frac{\partial u}{\partial x} = \frac{[(x^2-y^2+1)^2+4x^2y^2][3x^2+y^2+1]-[x^3+xy^2+x][2(x^2-y^2+1)\cdot 2x+8xy^2]}{[(x^2-y^2+1)^2+4x^2y^2]^2}$$

$$\frac{\partial v}{\partial y} = \frac{[(x^2-y^2+1)^2+4x^2y^2](1-x^2-3y^2)-(y-x^2y-y^3)[2(x^2-y^2+1)(-2y)+8x^2y]}{[(x^2-y^2+1)^2+4x^2y^2]^2}$$

Expanding numerator$\left(\frac{\partial u}{\partial x}\right) = -x^6+x^4y^2+x^4-x^2y^4+10x^2y^2-x^2-y^6+y^4+y^2-1$

Expanding numerator$\left(\frac{\partial v}{\partial y}\right) = -x^6+x^4y^2+x^4-x^2y^4+10x^2y^2-x^2-y^6+y^4+y^2-1$

$$\frac{\partial u}{\partial y} = \frac{[(x^2-y^2+1)^2+4x^2y^2](2xy)-(x^3+xy^2+x)[2(x^2-y^2+1)(-2y)+8x^2y]}{[(x^2-y^2+1)^2+4x^2y^2]^2}$$

$$\frac{\partial v}{\partial x} = \frac{[(x^2-y^2+1)^2+4x^2y^2](-2xy)-(y-x^2y-y^3)[2(x^2-y^2+1)(2x)+8xy^2]}{[(x^2-y^2+1)^2+4x^2y^2]^2}$$

Expanding numerator$\left(\frac{\partial u}{\partial y}\right) = -2xy(x^4+2x^2y^2-2x^2+y^4+2y^2-3)$

Expanding numerator$\left(\frac{\partial v}{\partial x}\right) = 2xy(x^4+2x^2y^2-2x^2+y^4+2y^2-3)$

10 Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, $f(z)$ is <u>analytic</u> except at $z = \pm i$

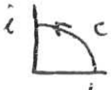
$\boxed{14.2.15}$ $f(z) = e^{\bar{z}} = \overline{e^x(\cos(y)+i\sin(y))} = e^x\cos(y) - ie^x\sin(y)$

$\frac{\partial u}{\partial x} = e^x\cos(y)$, $\frac{\partial v}{\partial y} = -e^x\cos(y)$, $\frac{\partial u}{\partial y} = -e^x\sin(y)$, $\frac{\partial v}{\partial x} = -e^x\sin(y)$

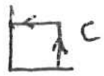
5 Since $u_x \neq v_y$ and $v_x \neq -u_y$, $f(z)$ is not analytic

$\boxed{14.2.24}$ $f(z) = \frac{y-ix}{x^2+y^2}$, $u_x = -\frac{2xy}{(x^2+y^2)^2}$, $v_y = \frac{2xy}{(x^2+y^2)^2}$, $u_y = \frac{x^2-y^2}{(x^2+y^2)^2}$, $v_x = \frac{x^2-y^2}{(x^2+y^2)^2}$

5 Since $u_x \neq v_y$ and $v_x \neq -u_y$, $f(z)$ is not analytic.

14.3.6 a. $\int_1^i z\,dz$ along  $z = e^{i\theta}$ $0 \le \theta \le \frac{\pi}{2}$

$$\oint_C z\,dz = \int_0^{\pi/2} e^{i\theta}\, i e^{i\theta} d\theta = i\int_0^{\pi/2} e^{2i\theta} d\theta = i\,\frac{e^{2i\theta}}{2i}\Big|_0^{\pi/2} = \tfrac{1}{2}(-1-1) = -1$$

b. along 

10 $$\oint_C z\,dz = \int_0^1 (1+iy)\,i\,dy + \int_1^0 (x+i)\,dx = \left(iy - \frac{y^2}{2}\right)\Big|_0^1 + \left(\frac{x^2}{2} + ix\right)\Big|_1^0 = i - \tfrac{1}{2} - \left(\tfrac{1}{2} + i\right) = -1$$

14.7.1. Evaluate:

$$\int_0^{2\pi} \frac{1}{13 + 5\sin\theta}\, d\theta.$$

Let $z = e^{i\theta}$. Then we have

$$dz = ir^{i\theta} d\theta \implies d\theta = \frac{1}{iz} dz.$$

By definition of $\sin\theta$, we have

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i}.$$

Let I be the integral

$$I = \int_0^{2\pi} \frac{1}{13 + 5\sin\theta}\, d\theta.$$

As θ runs through $[0, 2\pi]$, $z = e^{i\theta}$ traces the unit circle C in the counterclockwise direction. Thus we can make substitutions and turn I into a contour integral:

$$\begin{aligned} I &= \oint_C \frac{1}{13 + \frac{5}{2i}(z - 1/z)} \cdot \frac{1}{iz}\, dz \\ &= 2\oint_C \frac{1}{5z^2 + 26iz - 5}\, dz \\ &= 2\oint_C \frac{1}{(z+5i)(5z+i)}\, dz. \end{aligned}$$

This integrand has singularities at $z = -5i$ and $z = -i/5$; only $z = -i/5$ lies inside the unit circle C. Thus, we must find the residue of the integrand at $z = -i/5$:

$$R\left(-\frac{i}{5}\right) = \lim_{z \to -i/5} \left(z + \frac{i}{5}\right) \cdot \frac{1}{(z+5i)(5z+i)} = \lim_{z \to -i/5} \frac{1}{5(z+5i)} = \frac{1}{25i - i} = -\frac{i}{24}.$$

By the Residue Theorem,

$$\oint_C \frac{1}{(z+5i)(5z+i)} = 2\pi i \cdot R\left(-\frac{i}{5}\right) = \frac{\pi}{12}.$$

Thus,

$$I = 2\oint_C \frac{1}{(z+5i)(5z+i)} = \frac{\pi}{6},$$

so

10 $$\boxed{\int_0^{2\pi} \frac{1}{13 + 5\sin\theta}\, d\theta = \frac{\pi}{6}.}$$

Wolfram Alpha verifies this result.

14.7.10. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\, dx.$$

For $r > 0$, let

$$S_r = \{re^{i\theta} \in \mathbb{C} \mid \theta \in [0, \pi]\}$$

be the semicircle of radius r in the first and second quadrants of the plane and

$$I_r = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = 0,\ |z| \le r\}$$

be the real interval $[-r, r]$ embedded in the complex plane. Then $C_r = S_r \cup I_r$ is a simple loop in $\mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = \frac{1}{z^2+4z+5}.$$

Then

$$\begin{aligned}
\oint_{C_r} f(z)\, dz &= \int_{I_r} f(z)\, dz + \int_{S_r} f(z)\, dz \\
&= \int_{-r}^{r} f(x)\, dx + \int_0^{\pi} f(re^{i\theta}) ire^{i\theta}\, d\theta \\
&= \int_{-r}^{r} \frac{1}{x^2+4x+5}\, dx + \int_0^{\pi} \frac{ire^{i\theta}}{r^2e^{2i\theta}+4re^{i\theta}+5}\, d\theta.
\end{aligned}$$

As $r \to \infty$ we have

$$\begin{aligned}
\lim_{r\to\infty}\left[\oint_{C_r} f(z)\, dz\right] &= \lim_{r\to\infty}\left[\int_{-r}^{r} \frac{1}{x^2+4x+5}\, dx\right] + \lim_{r\to\infty}\left[\int_0^{\pi} \frac{ire^{i\theta}}{r^2e^{2i\theta}+4re^{i\theta}+5}\, d\theta\right] \\
&= \int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\, dx + \int_0^{\pi}\left[\lim_{r\to\infty} \frac{ire^{i\theta}}{r^2e^{2i\theta}+4re^{i\theta}+5}\right] d\theta \\
&= \int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\, dx + \int_0^{\pi} 0\, d\theta = \int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\, dx.
\end{aligned}$$

By the Residue Theorem, we also have

$$\lim_{r\to\infty}\left[\oint_{C_r} f(z)\, dz\right] = \lim_{r\to\infty}\left[2\pi i \sum_i R(z_i)\right]$$

where z_i is a singularity of $f(z)$ inside C_r. The singularities of f are the roots of the polynomial x^2+4x+5, which are $z_\pm = -2 \pm i$. The singularity $z_- - 2 - i$ is not inside the curve C_r for any $r > 0$, but for r large enough, $z_+ = -2+i$ is inside C_r. Thus we have

$$\lim_{r\to\infty}\left[\oint_{C_r} f(z)\, dz\right] = 2\pi i R(-2+i).$$

That is,

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\, dx = 2\pi i R(-2+i).$$

Now we calculate the residue at $z_+ = -2+i$:

$$\begin{aligned}
R(-2+i) &= \lim_{z\to -2+i}\left[(z+2-i)\frac{1}{z^2+4z+5}\right] \\
&= \lim_{z\to -2+i}\left[(z+2-i)\frac{1}{(z+2-i)(z+2+i)}\right] \\
&= \lim_{z\to -2+i}\left[\frac{1}{z+2+i}\right] = \frac{1}{2i} = -\frac{i}{2}.
\end{aligned}$$

10 Thus,

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\, dx = 2\pi i R(-2+i) = 2\pi i\left(-\frac{i}{2}\right) = \pi.$$

So,

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{x^2+4x+5}\, dx = \pi.}$$

Wolfram Alpha agrees.

14.7.12. Evaluate

$$\int_0^\infty \frac{x^2}{x^4+16}\,dx.$$

Let $f:\mathbb{R}\to\mathbb{R}$ be the function given by $f(x)=x^2/(x^4+16)$. Then f is an even function, so that for $a>0$ we have

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Now consider the extension of f into the complex plane. For some $r>0$, let

$$S_r = \{re^{i\theta}\in\mathbb{C}\mid \theta\in[0,\pi]\}$$

be the semicircle of radius r in the first and second quadrants of the plane and

$$I_r = \{z\in\mathbb{C}\mid \operatorname{Im}(z)=0,\ |z|\le r\}$$

be the real interval $[-r,r]$ embedded in the complex plane. Then $C_r = S_r\cup I_r$ is a simple loop in $\mathbb{C}$. Then

$$\begin{aligned}\oint_{C_r} f(z)\,dz &= \int_{I_r} f(z)\,dz + \int_{S_r} f(z)\,dz\\ &= \int_{-r}^{r} f(x)\,dx + \int_0^\pi f(re^{i\theta})ire^{i\theta}\,d\theta\\ &= 2\int_0^r \frac{x^2}{x^4+16}\,dx + \int_0^\pi \frac{ir^3e^{3i\theta}}{r^4e^{4i\theta}+16}\,d\theta.\end{aligned}$$

As $r\to\infty$ we have

$$\begin{aligned}\lim_{r\to\infty}\left[\oint_{C_r} f(z)\,dz\right] &= \lim_{r\to\infty}\left[2\int_0^r \frac{x^2}{x^4+16}\,dx\right] + \lim_{r\to\infty}\left[\int_0^\pi \frac{ir^3e^{3i\theta}}{r^4e^{4i\theta}+16}\,d\theta\right]\\ &= 2\int_0^\infty \frac{x^2}{x^4+16}\,dx + \int_0^\pi \left[\lim_{r\to\infty}\frac{ir^3e^{3i\theta}}{r^4e^{4i\theta}+16}\right]d\theta\\ &= 2\int_0^\infty \frac{x^2}{x^4+16}\,dx + \int_0^\pi 0\,d\theta = 2\int_0^\infty \frac{x^2}{x^4+16}\,dx.\end{aligned}$$

By the Residue Theorem, we also have

$$\lim_{r\to\infty}\left[\oint_{C_r} f(z)\,dz\right] = \lim_{r\to\infty}\left[2\pi i\sum_i R(z_i)\right]$$

where z_i is a singularity of $f(z)$ inside C_r. The singularities of f are the roots of the polynomial x^4+16, which are $z_{\pm\pm}=\pm\sqrt{2}\pm i\sqrt{2}$. The singularities $z_{\pm-}=\pm\sqrt{2}-i\sqrt{2}$ are not inside the curve C_r for any $r>0$, but for r large enough, $z_{\pm+}=\pm\sqrt{2}+i\sqrt{2}$ are inside C_r. Thus we have

$$\lim_{r\to\infty}\left[\oint_{C_r} f(z)\,dz\right] = 2\pi i\Big[R\big(\sqrt{2}+i\sqrt{2}\big) + R\big(-\sqrt{2}+i\sqrt{2}\big)\Big].$$

That is,

$$\int_0^\infty \frac{x^2}{x^4+16}\,dx = \pi i\Big[R\big(\sqrt{2}+i\sqrt{2}\big) + R\big(-\sqrt{2}+i\sqrt{2}\big)\Big].$$

Now we calculate the residue at $z_{++} = \sqrt{2} + i\sqrt{2}$:

$$\begin{aligned}
R\left(\sqrt{2}+i\sqrt{2}\right) &= \lim_{z\to\sqrt{2}+i\sqrt{2}} \left[\left(z-\sqrt{2}-i\sqrt{2}\right)\frac{z^2}{z^4+16}\right] \\
&= \lim_{z\to\sqrt{2}+i\sqrt{2}} \left[\frac{z^2}{\left(z-\sqrt{2}+i\sqrt{2}\right)\left(z+\sqrt{2}-i\sqrt{2}\right)\left(z+\sqrt{2}+i\sqrt{2}\right)}\right] \\
&= \frac{\left(\sqrt{2}+i\sqrt{2}\right)^2}{\left(2i\sqrt{2}\right)\left(2\sqrt{2}\right)2\left(\sqrt{2}+i\sqrt{2}\right)} \\
&= \frac{\sqrt{2}+i\sqrt{2}}{16i}.
\end{aligned}$$

Next, the residue at at $z_{-+} = -\sqrt{2} + i\sqrt{2}$ is

$$\begin{aligned}
R\left(-\sqrt{2}+i\sqrt{2}\right) &= \lim_{z\to-\sqrt{2}+i\sqrt{2}} \left[\left(z+\sqrt{2}-i\sqrt{2}\right)\frac{z^2}{z^4+16}\right] \\
&= \lim_{z\to-\sqrt{2}+i\sqrt{2}} \left[\frac{z^2}{\left(z-\sqrt{2}+i\sqrt{2}\right)\left(z-\sqrt{2}-i\sqrt{2}\right)\left(z+\sqrt{2}+i\sqrt{2}\right)}\right] \\
&= \frac{\left(-\sqrt{2}+i\sqrt{2}\right)^2}{2\left(-\sqrt{2}+i\sqrt{2}\right)\left(-2\sqrt{2}\right)\left(2i\sqrt{2}\right)} \\
&= \frac{\sqrt{2}-i\sqrt{2}}{16i}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\infty \frac{x^2}{x^4+16}\,dx &= \pi i\left[R\left(\sqrt{2}+i\sqrt{2}\right)+R\left(-\sqrt{2}+i\sqrt{2}\right)\right] \\
&= 2\pi i\left(\frac{\sqrt{2}+i\sqrt{2}}{16i}+\frac{\sqrt{2}-i\sqrt{2}}{16i}\right) \\
&= 2\pi i\left(\frac{2\sqrt{2}}{16i}\right) \\
&= \frac{\pi\sqrt{2}}{8}.
\end{aligned}$$

So,

$$\boxed{\int_0^\infty \frac{x^2}{x^4+16}\,dx = \frac{\pi\sqrt{2}}{8}.}$$

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Wolfram Alpha answers $\frac{\pi}{4\sqrt{2}}$, which is the same thing.

14.7.16. Evaluate

$$\int_0^\infty \frac{x\sin x}{9x^2+4}\,dx$$

For $r > 0$, let

$$S_r = \{re^{i\theta} \in \mathbb{C} \mid \theta \in [0,\pi]\}$$

be the semicircle of radius r in the first and second quadrants of the plane and

$$I_r = \{z \in \mathbb{C} \mid \operatorname{Im}(z) = 0,\ |z| \le r\}$$

be the real interval $[-r, r]$ embedded in the complex plane. Then $C_r = S_r \cup I_r$ is a simple loop in $\mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) = \frac{ze^{iz}}{9z^2+4}.$$

Then

$$\begin{aligned}
\oint_{C_r} f(z)\,dz &= \int_{I_r} f(z)\,dz + \int_{S_r} f(z)\,dz \\
&= \int_{-r}^{r} f(x)\,dx + \int_0^\pi f\big(re^{i\theta}\big)ire^{i\theta}\,d\theta \\
&= \int_{-r}^{r} \frac{xe^{ix}}{9x^2+4}\,dx + \int_0^\pi \frac{ir^2e^{2i\theta}e^{ire^{i\theta}}}{9r^2e^{2i\theta}+4}\,d\theta \\
&= \int_{-r}^{r} \frac{xe^{ix}}{9x^2+4}\,dx + \int_0^\pi \frac{ir^2e^{-2\theta re^{i\theta}}}{9r^2e^{2i\theta}+4}\,d\theta.
\end{aligned}$$

As $r \to \infty$, $e^{-2\theta re^{i\theta}}$ decays to 0 faster than $ir^2/\big(9r^2e^{2i\theta}+4\big)$ converges. Thus, we have

$$\begin{aligned}
\lim_{r\to\infty}\left[\oint_{C_r} f(z)\,dz\right] &= \lim_{r\to\infty}\left[\int_{-r}^{r} \frac{xe^{ix}}{9x^2+4}\,dx\right] + \lim_{r\to\infty}\left[\int_0^\pi \frac{ir^2e^{-2\theta re^{i\theta}}}{9r^2e^{2i\theta}+4}\,d\theta\right] \\
&= \int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2+4}\,dx + \int_{-\infty}^{\pi}\left[\lim_{r\to\infty}\frac{ir^2e^{-2\theta re^{i\theta}}}{9r^2e^{2i\theta}+4}\right]d\theta \\
&= \int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2+4}\,dx + \int_{-\infty}^{\pi} 0\,d\theta = \int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2+4}\,dx.
\end{aligned}$$

By the Residue Theorem, we also have

$$\lim_{r\to\infty}\left[\oint_{C_r} f(z)\,dz\right] = \lim_{r\to\infty}\left[2\pi i \sum_i R(z_i)\right]$$

where z_i is a singularity of $f(z)$ inside C_r. The singularities of f are the roots of the polynomial $9x^2+4$, which are $z_\pm = \pm\frac{2}{3}i$. The singularity $z_- = -\frac{2}{3}i$ is not inside the curve C_r for any $r > 0$, but for r large enough, the singularity $z_+ = \frac{2}{3}i$ is inside C_r. Thus we have

$$\lim_{r\to\infty}\left[\oint_{C_r} f(z)\,dz\right] = 2\pi iR\left(\frac{2}{3}i\right) \implies \int_{-\infty}^{\infty}\frac{xe^{ix}}{9x^2+4}\,dx = 2\pi iR\left(\frac{2}{3}i\right).$$

Now we calculate the residue at the singularity $z_+ = \frac{2}{3}i$:

$$\begin{aligned} R\left(\frac{2}{3}i\right) &= \lim_{z \to \frac{2}{3}i} \left[\left(z - \frac{2}{3}i\right) \frac{ze^{iz}}{9z^2+4}\right] \\ &= \lim_{z \to \frac{2}{3}i} \left[\frac{ze^{iz}}{9z+6i}\right] \\ &= \frac{\frac{2}{3}ie^{-2/3}}{12i} = \frac{1}{18e^{2/3}}. \end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2+4}\,dx = 2\pi i R\left(\frac{2}{3}i\right) = 2\pi i \cdot \frac{1}{18e^{2/3}} = \frac{\pi}{9e^{2/3}}i.$$

Since $e^{ix} = \cos x + i \sin x$, by linearity of the integral we have

$$\begin{aligned} \frac{\pi}{9e^{2/3}}i &= \int_{-\infty}^{\infty} \frac{xe^{ix}}{9x^2+4}\,dx \\ &= \int_{-\infty}^{\infty} \frac{x[\cos x + i \sin x]}{9x^2+4}\,dx \\ &= \int_{-\infty}^{\infty} \frac{x \cos x}{9x^2+4}\,dx + i\int_{-\infty}^{\infty} \frac{x \sin x}{9x^2+4}\,dx. \end{aligned}$$

Since the real and imaginary parts of the above equations must be equal, we have

$$\int_{-\infty}^{\infty} \frac{x \sin x}{9x^2+4}\,dx = \frac{\pi}{9e^{2/3}}.$$

Furthermore, $x \sin(x)/(9x^2+4)$ is an even function, so that

$$\int_{0}^{\infty} \frac{x \sin x}{9x^2+4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin x}{9x^2+4}\,dx = \frac{1}{2} \cdot \frac{\pi}{9e^{2/3}} = \frac{\pi}{18e^{2/3}}.$$

Therefore,

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$$\boxed{\int_{0}^{\infty} \frac{x \sin x}{9x^2+4}\,dx = \frac{\pi}{18e^{2/3}}.}$$

Wolfram Alpha confirms.

14.7.30. **a.** By the method of Example 2 evaluate

$$\int_0^\infty \frac{1}{1+x^4}\, dx.$$

b. Evaluate the same integral using tables or computer to get the indefinite integral; unless you are very careful you may get zero. Explain why.

c. Make the change of variables $u = x^4$ in the integral and evaluate the u integral using (7.4).

a. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 1/(1+x^4)$. Then f is an even function, so that for $a > 0$ we have

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx.$$

Now consider the extension of f into the complex plane. For some $r > 0$, let

$$S_r = \{re^{i\theta} \in \mathbb{C} \mid \theta \in [0, \pi]\}$$

be the semicircle of radius r in the first and second quadrants of the plane and

$$I_r = \{z \in \mathbb{C} \mid \mathrm{Im}(z) = 0,\ |z| \le r\}$$

be the real interval $[-r, r]$ embedded in the complex plane. Then $C_r = S_r \cup I_r$ is a simple loop in $\mathbb{C}$. Then

$$\begin{aligned}\oint_{C_r} f(z)\, dz &= \int_{I_r} f(z)\, dz + \int_{S_r} f(z)\, dz \\ &= \int_{-r}^{r} f(x)\, dx + \int_0^\pi f(re^{i\theta}) ire^{i\theta}\, d\theta \\ &= 2\int_0^r \frac{1}{1+x^4}\, dx + \int_0^\pi \frac{ire^{i\theta}}{1+r^4e^{4i\theta}}\, d\theta.\end{aligned}$$

As $r \to \infty$ we have

$$\begin{aligned}\lim_{r\to\infty}\left[\oint_{C_r} f(z)\, dz\right] &= \lim_{r\to\infty}\left[2\int_0^r \frac{1}{1+x^4}\, dx\right] + \lim_{r\to\infty}\left[\int_0^\pi \frac{ire^{i\theta}}{1+r^4e^{4i\theta}}\, d\theta\right] \\ &= 2\int_0^\infty \frac{1}{1+x^4}\, dx + \int_0^\pi \left[\lim_{r\to\infty} \frac{ire^{i\theta}}{1+r^4e^{4i\theta}}\right] d\theta \\ &= 2\int_0^\infty \frac{1}{1+x^4}\, dx + \int_0^\pi 0\, d\theta = 2\int_0^\infty \frac{1}{1+x^4}\, dx.\end{aligned}$$

By the Residue Theorem, we also have

$$\lim_{r\to\infty}\left[\oint_{C_r} f(z)\, dz\right] = \lim_{r\to\infty}\left[2\pi i \sum_i R(z_i)\right]$$

where z_i is a singularity of $f(z)$ inside C_r. The singularities of f are the roots of the polynomial $1+x^4$, which are $z_{\pm\pm} = \pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$. The singularities $z_{\pm-} = \pm\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$ are not inside the curve C_r for any $r > 0$, but for r large enough, $z_{\pm+} = \pm\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ are inside

C_r. Thus we have

$$\lim_{r\to\infty}\left[\oint_{C_r} f(z)\,dz\right] = 2\pi i\left[R\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)+R\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\right].$$

That is,

$$\int_0^\infty \frac{1}{1+x^4}\,dx = \pi i\left[R\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)+R\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\right].$$

Now we calculate the residue at $z_{++}=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$:

$$\begin{aligned}
R\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) &= \lim_{z\to\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}\left[\left(z-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\frac{1}{1+z^4}\right]\\
&= \lim_{z\to\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}\left[\frac{1}{\left(z-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\left(z+\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\left(z+\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}\right]\\
&= \frac{1}{\left(2i\frac{\sqrt{2}}{2}\right)\left(2\frac{\sqrt{2}}{2}\right)2\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)} = \frac{1}{-2\sqrt{2}+2i\sqrt{2}}.
\end{aligned}$$

Next, the residue at at $z_{-+}=-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ is

$$\begin{aligned}
R\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) &= \lim_{z\to-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}\left[\left(z+\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\frac{1}{1+z^4}\right]\\
&= \lim_{z\to-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}}\left[\frac{z^2}{\left(z-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\left(z-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)\left(z+\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)}\right]\\
&= \frac{1}{2\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\left(-2\frac{\sqrt{2}}{2}\right)\left(2i\frac{\sqrt{2}}{2}\right)} = \frac{1}{2\sqrt{2}+2i\sqrt{2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^4}\,dx &= \pi i\left[R\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)+R\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)\right]\\
&= \pi i\left(\frac{1}{-2\sqrt{2}+2i\sqrt{2}}+\frac{1}{2\sqrt{2}+2i\sqrt{2}}\right)\\
&= \frac{\pi}{2\sqrt{2}}\left(\frac{1}{1+i}+\frac{1}{1-i}\right)\\
&= \frac{\pi}{2\sqrt{2}}\cdot\frac{1+i+1-i}{2} = \frac{\pi}{2\sqrt{2}}\cdot\frac{2}{2} = \frac{\pi}{2\sqrt{2}}.
\end{aligned}$$

So,

$$\boxed{\int_0^\infty \frac{1}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}}.}$$

b. According to Wolfram Alpha, the integral in part **a.** is equal to $\pi/(2\sqrt{2})$, which is what we found. Also according to Wolfram Alpha, the indefinite integral

$$\int \frac{1}{1+x^4}\,dx$$

is equal to

$$\frac{1}{4\sqrt{2}}\Big[-\ln\left(x^2-\sqrt{2}x+1\right)+\ln\left(x^2+\sqrt{2}x+1\right)$$
$$-2\tan^{-1}\left(1-\sqrt{2}x\right)+2\tan^{-1}\left(\sqrt{2}x+1\right)\Big]+K.$$

c. Equation (7.5) says

$$\int_0^\infty \frac{r^{p-1}}{1+r}\,dr=\frac{\pi}{\sin(\pi p)}$$

Let $u=x^4$. Then $du=4x^3dx$, so that

$$dx=\frac{1}{4}x^{-3}du=\frac{1}{4}u^{-3/4}du=\frac{1}{4}u^{1/4-1}du.$$

Then by (7.5) we have

$$\int_0^\infty \frac{1}{1+x^4}=\frac{1}{4}\int_0^\infty \frac{u^{1/4-1}}{1+u}\,du=\frac{1}{4}\cdot\frac{\pi}{\sin(\pi/4)}=\frac{1}{4}\cdot\frac{\pi}{\sqrt{2}/2}=\boxed{\frac{\pi}{2\sqrt{2}}},$$

as expected.

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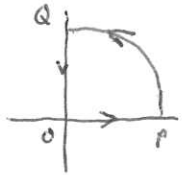
$\boxed{14.7.45}$



$f(z) = z^3 + z^2 + z + 4 = 0$

Along $\overrightarrow{OP}$ $f(z)$ is real, so $\Delta \arg(f(z)) = 0$.
There is no change in argument along $\overrightarrow{OP}$.

For P sufficiently large, z^3 dominates expression, $f(z) \sim z^3 = R^3 e^{i3\theta}$
$\lim_{R\to\infty} \Delta \arg(f(z)) \approx 3\pi/2$. Along $\overrightarrow{QO}$ $z = iy$ $\therefore f(iy) = -i(y^3 - y) + 4 - y^2$
$\arg f(iy) = \arctan\left(\frac{y - y^3}{4 - y^2}\right)$. The image of $\overrightarrow{QO}$ starts in the 3^{rd} quadrant with argument near $3\pi/2$. As y decreases, the image stays in the 3^{rd} quadrant until $y = 2$ when the real part becomes positive crossing into the 4^{th} quadrant. The imaginary part becomes positive at $y = 1$, then returns to the positive x-axis as $y \to 0$. Thus $\Delta \arg(f(z)) = \pi/2$ along $\overrightarrow{QO}$.

$N = \frac{1}{2\pi} \Delta_c \arg(f(z)) = \frac{1}{2\pi}(0 + 3\pi/2 + \pi/2) = 1.$

One root in 1^{st} quadrant, conjugate root in 4^{th} quadrant and real root on the negative axis.

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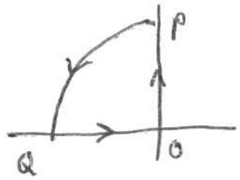
$\boxed{14.7.50}$



$f(z) = z^4 + z^3 + 4z^2 + 2z + 3 = 0$

Along $\overrightarrow{QO}$, $f(z)$ is real, so $\Delta \arg(f(z)) = 0$.

Along $\overrightarrow{OP}$, $z = iy$, so $\arg(f(z)) = \arctan\left(\frac{-y^3 + 2y}{y^4 - 4y^2 + 3}\right)$. At 0, the image begins at 3. As y increases, the image curve traces a path through the 1^{st} quadrant until $y = 1$, then the image crosses to the 2^{nd} quadrant. At $y = \sqrt{2}$, the image crosses from the 2^{nd} quadrant into the 3^{rd} quadrant. At $y = \sqrt{3}$, the image moves from the 3^{rd} quadrant to the 4^{th} quadrant. As y increases, it stays in the 4^{th} quadrant and $\lim_{y\to\infty} \arg(f(z)) \to 0$, completing one encirclement of the origin.

Along $\overrightarrow{PQ}$, $f(z) \sim z^4 = R^4 e^{i4\theta}$. As θ traverses $\pi/2$ radians ($\frac{\pi}{2} \le \theta \le \pi$), the image of $f(z)$ has $\Delta \arg(f(z)) \approx 2\pi$. Thus, the total change in argument gives $N = \frac{1}{2\pi} \Delta_c \arg(f(z)) = \frac{1}{2\pi}(0 + 2\pi + 2\pi) = 2$.

It follows there are two roots in the 2^{nd} quadrant, so by the form of $f(z)$ there must also be two roots in the 3^{rd} quadrant (complex conjugates). No roots appear in the 1^{st} + 4^{th} quadrants.

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