Spring 2012 TH 1 Math 342B

① a. $A = \begin{pmatrix} 0 & -a & b \\ -a & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ e.v.'s $\det \begin{vmatrix} -\lambda & -a & b \\ -a & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2 - a^2) = 0$

$a \neq \pm 2$

e.v. $\lambda_1 = 2$, $\begin{cases} -2\xi_1 - a\xi_2 + b\xi_3 = 0 \\ -a\xi_1 - 2\xi_2 = 0 \end{cases}$ Let $\xi_3 = 1$, $\xi_2 = -\frac{a}{2}\xi_1$, $\xi_1 = \frac{-2b}{a^2-4}$

e.f. $\vec{\xi}_1 = \left[-\frac{2b}{a^2-4}, \frac{ab}{a^2-4}, 1\right]^T$

e.v. $\lambda_2 = a$, $\begin{cases} -a\xi_1 - a\xi_2 + b\xi_3 = 0 \\ -a\xi_1 - a\xi_2 = 0 \\ (2-a)\xi_3 = 0 \end{cases}$ $\xi_3 = 0$, $\xi_1 = -\xi_2$ e.f. $\vec{\xi}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

e.v. $\lambda_3 = -a$, $\begin{cases} a\xi_1 - a\xi_2 + b\xi_3 = 0 \\ -a\xi_1 + a\xi_2 = 0 \\ (2+a)\xi_3 = 0 \end{cases}$ $\xi_3 = 0$, $\xi_1 = \xi_2$ e.f. $\vec{\xi}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$a \neq \pm 2$, 3 distinct e.v.'s & e.f.'s

If $a = 2$, $\lambda_1 = 2$ alg. mult. $= 2$ $\quad -2\xi_1 - 2\xi_2 - b\xi_3 = 0$, $-2\xi_1 - 2\xi_2 = 0$ $\quad \xi_1 = -\xi_2$ $\quad \xi_3 = 0$ unless $b = 0$

geo. mult. $= 1$ if $b \neq 0$ e.f. $\vec{\xi}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

if $b = 0$ geo. mult. $= 2$, e.f. $\vec{\xi}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\xi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$, e.f. $\vec{\xi}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

If $a = -2$, $\lambda_1 = 2$ alg. mult. $= 2$ geo. mult. $= 2$ e.f. $\vec{\xi}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -2$, e.f. $\vec{\xi}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

---

b. Take $\vec{v}_1 = \begin{pmatrix} 0 & -a_1 & b_1 \\ -a_1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 & -a_2 & b_2 \\ -a_2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\vec{v}_1 + \vec{v}_2 = \begin{pmatrix} 0 & -(a_1+a_2) & (b_1+b_2) \\ -(a_1+a_2) & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$\vec{v}_1 + \vec{v}_2 \notin$ vector space, not closed or fails to have additive identity element (the zero vector)

---

② $\Gamma(-\frac{7}{2}) = \dfrac{\Gamma(-\frac{5}{2})}{-\frac{7}{2}} = \dfrac{\Gamma(-\frac{3}{2})}{(-\frac{7}{2})(-\frac{5}{2})} = \dfrac{\Gamma(-\frac{1}{2})}{(-\frac{7}{2})(-\frac{5}{2})(-\frac{3}{2})} = \dfrac{\Gamma(\frac{1}{2})}{(-\frac{7}{2})(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})} = \dfrac{16\sqrt{\pi}}{105}$

(3) $$\int_0^{\pi} \cos^6(\theta)\,d\theta = 2\int_0^{\pi/2} \cos^6(\theta)\,d\theta = B(\tfrac{1}{2},\tfrac{7}{2}) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{7}{2})}{\Gamma(4)} = \frac{\sqrt{\pi}\,\Gamma(\frac{7}{2})}{3!}$$

$2p-1=0$
$2q-1=6$

$$\Gamma(\tfrac{7}{2}) = \tfrac{5}{2}\Gamma(\tfrac{5}{2}) = \tfrac{5}{2}\cdot\tfrac{3}{2}\Gamma(\tfrac{3}{2}) = \tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \tfrac{15}{8}\sqrt{\pi}$$

$$\int_0^{\pi} \cos^6(\theta)\,d\theta = \frac{\sqrt{\pi}}{6}\cdot\frac{15}{8}\sqrt{\pi} = \frac{5}{6}\pi$$

---

(4) $$\int_0^3 \frac{dx}{\sqrt{(16-x^2)(9-x^2)}} = \int_0^1 \frac{3\,du}{\sqrt{(16-9u^2)(9-9u^2)}} = \frac{1}{4}\int_0^1 \frac{du}{\sqrt{(1-(\frac{3}{4})^2u^2)(1-u^2)}}$$

$u = x/3 \quad du = dx/3$

$$= \frac{1}{4}K(\tfrac{3}{4})$$

---

(5) Least squares best fit to order 5 $\quad \sin(x) \approx \sum_{n=0}^{5} a_n P_n(x)$

By orthogonality $\quad a_n = \frac{2n+1}{2}\int_{-1}^{1} P_n(x)\sin(x)\,dx \quad , n = 0,\ldots,5$

Since $\sin(x)$ is odd $\quad a_0 = a_2 = a_4 = 0$

$$a_1 = \frac{3}{2}\int_{-1}^{1} x\sin(x)\,dx = 3\sin(1) - 3\cos(1) \approx 0.9305$$

$$a_3 = \frac{7}{2}\int_{-1}^{1} \frac{1}{2}(5x^3-3x)\sin(x)\,dx = 98\cos(1) - 63\sin(1) \approx -0.063046$$

$$a_5 = \frac{11}{2}\int_{-1}^{1} \frac{1}{8}(63x^5-70x^3+15x)\sin(x)\,dx = 5940\sin(1) - 9251\cos(1) \approx 0.001018$$

$$\sin(x) \approx 0.9305\,P_1(x) - 0.063046\,P_3(x) + 0.001018\,P_5(x)$$

---

(6) Consider $\quad xy'' - y' + y = 0$.

Put in form $\quad y'' + \frac{1-2a}{x}y' + \left[(bcx^{c-1})^2 + \frac{a^2-p^2c^2}{x^2}\right]y = 0$

$$y'' - \frac{1}{x}y' + \frac{1}{x}y = 0 \Rightarrow 1-2a=-1,\quad a^2-p^2c^2=0,\quad (bc)^2=1 \quad 2(c-1)=-1$$

$a=1,\ c=\frac{1}{2},\ b=\pm 2,\ p=\pm 2 \quad$ without loss of generality take $b=2,\ p=2$

Soln. $\quad y(x) = A\,x\,J_2(2\sqrt{x}) + B\,x\,N_2(2\sqrt{x})$

(7) Consider $y'' - xy' - y = 0$ with $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ $\quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

a_0 + a_1 arb. $\quad x^0$: $2a_2 - a_0 = 0$ $\quad n \geq 1$: $(n+2)(n+1) a_{n+2} = (n+1) a_n$

Recurrence Relation: $a_{n+2} = \frac{a_n}{n+2}$

$a_2 = \frac{1}{2} a_0$, $a_4 = \frac{1}{4 \cdot 2} a_0$, $a_{2n} = \frac{1}{2n(2n-2)\cdots 4\cdot 2} a_0 = \frac{1}{2^n \cdot n!} a_0$

$a_3 = \frac{1}{3} a_1$, $a_5 = \frac{1}{5\cdot 3} a_1$, $a_{2n+1} = \frac{1}{(2n+1)(2n-1)\cdots 3} a_1 = \frac{2^n n!}{(2n+1)!} a_1$

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + a_1 \sum_{n=0}^{\infty} \frac{2^n n! \, x^{2n+1}}{(2n+1)!}$$

---

(8) Consider $x^2 y'' + 3xy' + (1+x)y = 0$.

$\lim_{x\to 0} \frac{x(3x)}{x^2} = 3$ and $\lim_{x\to 0} \frac{x^2(1+x)}{x^2} = 1$. Thus, $x=0$ is a regular sing. pt.

Method of Frobenius: $y = \sum_{n=0}^{\infty} a_n x^{n+s}$, $y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$, $y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$

$$\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s} + \sum_{n=0}^{\infty} 3(n+s) a_n x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s} + \sum_{n=1}^{\infty} a_{n-1} x^{n+s} = 0$$

$n=0$: $s(s-1) + 3s + 1 = s^2 + 2s + 1 = 0$ $\quad s = -1, -1$ $\quad$ Indicial Eqn.

Recurrence relation: $s=-1$ $\quad [(n-1)(n-2) + 3(n-1) + 1] a_n = -a_{n-1}$

$a_n = -\frac{a_{n-1}}{n^2}$ $\quad a_1 = -a_0$, $a_2 = \frac{1}{4} a_0$, $a_3 = -\frac{1}{9\cdot 4} a_0$, ...

$y_1(x) = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^n$ $\quad$ Since $s=-1$ repeated root

$y_2(x) = y_1(x) \ln(x) + x^{-1} \sum_{n=1}^{\infty} b_n x^n$ $\quad y_2' = y_1' \ln|x| + \frac{1}{x} y_1 + \sum_{n=2}^{\infty} (n-1) b_n x^{n-2}$

$y_2'' = y_1'' \ln|x| + \frac{2}{x} y_1' - \frac{1}{x^2} y_1 + \sum_{n=3}^{\infty} (n-1)(n-2) b_n x^{n-3}$

Substituting

$$\ln|x| \underbrace{\left(x^2 y_1'' + 3xy_1' + (1+x) y_1 \right)}_{=0} + 2xy_1' - y_1 + 3y_1 + \sum_{n=3}^{\infty} (n-1)(n-2) b_n x^{n-1} + 3\sum_{n=2}^{\infty} (n-1) b_n x^{n-1}$$
$$+ \sum_{n=1}^{\infty} b_n x^{n-1} + \sum_{n=1}^{\infty} b_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)\, b_{n+1} x^n + 3\sum_{n=1}^{\infty} n\, b_{n+1} x^n + \sum_{n=0}^{\infty} b_{n+1} x^n = -\sum_{n=1}^{\infty} b_n x^n - 2x y_1' + 2y_1$$

$$= -\sum_{n=1}^{\infty} b_n x^n - 2\left(\sum_{n=0}^{\infty} (n-1)\frac{(-1)^n x^{n-1}}{(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n-1}}{(n!)^2}\right)$$

$$= -\sum_{n=1}^{\infty} b_n x^n - 2\left(\sum_{n=0}^{\infty} (n+1)\frac{(-1)^{n+1} x^n}{((n+1)!)^2}\right)$$

x^0: $b_1 = 2$

x^1: $4b_2 = -b_1 - 1 \Rightarrow b_2 = -\frac{3}{4}$

Recurrence relation:

$$(n+1)^2 b_{n+1} = -b_n + \frac{2(-1)^n}{(n+1)!\,n!}, \qquad b_{n+1} = \frac{-(n+1)!\,n!\,b_n - 2(-1)^n}{(n+1)!\,n!\,(n+1)^2}$$

$b_3 = \frac{11}{108}$, $b_4 = -\frac{25}{3456}$, ...

$$\therefore y_2(x) = y_1(x)\ln|x| + 2 - \frac{3}{4}x + \frac{11}{108}x^2 - \frac{25}{3456}x^3 + \ldots$$

$$= y_1(x)\ln|x| - \frac{2}{x}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{(n!)^2} x^n \qquad \text{with } H_n = \left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + 1\right)$$