Math 531 HW # 7 Susan Guyler

7.3.1 b) Consider the heat equation in a two-dimensional rectangular region $0<x<L$, $0<y<H$

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

I.C. $u(x,y,0) = f(x,y)$

B.C: $\frac{\partial u}{\partial x}(0,y,t)=0$ $\frac{\partial u}{\partial x}(L,y,t)=0$

$\frac{\partial u}{\partial y}(x,0,t)=0$ $\frac{\partial u}{\partial y}(x,H,t)=0$

Solve the (IVP) and analyse the temperature as $t \to \infty$

Separation of variables: I am assuming k is constant

$u(x,y,t) = \phi(x)g(y)h(t)$

$$h'\phi g = k(\phi'' gh + g''\phi h)$$

dividing by $k\phi gh$

$$\frac{h'}{kh} = \frac{\phi''}{\phi} + \frac{g''}{g} \overset{set}{=} -\lambda$$

① $h' = -\lambda k h \Rightarrow h(t) = ce^{-k\lambda t}$

② $\frac{\phi''}{\phi} + \frac{g''}{g} = -\lambda$

$$\frac{\phi''}{\phi} = -\lambda - \frac{g''}{g} \overset{set}{=} -\nabla$$

(2a) $\phi'' = -\nabla\phi$ $\phi'(0)=0$ $\phi'(L)=0$

$\nabla = -\alpha^2 < 0$: done before only trivial solution

$\nabla = 0$

$\phi'' = 0$ $\phi(x) = c_1x + c_2$

$\phi'(0) = c_1 = 0$

$\nabla = \alpha^2 > 0$

$\phi'' = -\nabla \phi$

$\phi(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$

$\phi'(x) = -\alpha C_1 \sin(\alpha x) + \alpha C_2 \cos(\alpha x)$

$\phi'(0) = \alpha C_2 = 0 \Rightarrow C_2 = 0$

$\phi'(L) = -\alpha C_1 \sin(\alpha L) = 0$

$\sin(\alpha L) = 0$ for $\alpha = \frac{n\pi}{L}$

$\nabla_n = \frac{n^2\pi^2}{L^2} \quad \phi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$ for $n = 1, 2, 3, \ldots$

(2b) $-\lambda + \frac{g''}{g} = -\nabla_n \qquad g'(0) = 0 \quad g'(H) = 0$

$g'' = -(\lambda_n - \nabla_n) g \qquad$ say $\lambda_n - \nabla_n = \gamma_n$

$g'' = -\gamma g$

$\gamma = -\alpha^2 < 0$: only trivial solution

$\gamma = 0$

$g'' = 0 \qquad g(y) = C_1 y + C_2$

$g'(0) = C_1 = 0$

$\Rightarrow g(y) = C_2$ - constant

$\gamma_0 = 0 \qquad g_0(y) = 1$

$\gamma = \alpha^2 > 0$

$g'' = -\gamma g$

$g(y) = C_1 \cos(\alpha y) + C_2 \sin(\alpha y)$

$g'(y) = -\alpha C_1 \sin(\alpha y) + \alpha C_2 \cos(\alpha y)$

$g'(0) = \alpha C_2 = 0 \Rightarrow C_2 = 0$

$g'(H) = -\alpha C_1 \sin(\alpha H) = 0$

$\sin(\alpha H) = 0$ for $\alpha = \frac{m\pi}{H}$

$\Rightarrow \gamma_m = \frac{m^2\pi^2}{H^2} \quad g_m(y) = \cos\left(\frac{m\pi y}{H}\right)$ for $m = 1, 2, 3, \ldots$

$\lambda_{nm} = \gamma_m + \nabla_n = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{H^2}$

$h(t) = a_{nm} e^{-k\lambda_{nm} t}$

$$U_{mn}(x,y,t) = a_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)e^{-k\left(\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{H^2}\right)t}$$

superposition principle

$$* \; u(x,y,t) = A_{00} + \sum_{n=1}^{\infty} A_{n0}\cos\left(\frac{n\pi x}{L}\right)e^{-k\frac{n^2\pi^2}{L^2}t}$$
$$+ \sum_{m=1}^{\infty} A_{0m}\cos\left(\frac{m\pi y}{H}\right)e^{-k\frac{m^2\pi^2}{H^2}t}$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)e^{-k\left(\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{H^2}\right)t}$$

$$u(x,y,0) = f(x,y) = A_{00} + \sum_{n=1}^{\infty} A_{n0}\cos\left(\frac{n\pi x}{L}\right)$$
$$+ \sum_{m=1}^{\infty} A_{0m}\cos\left(\frac{m\pi y}{H}\right) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)A_{nm}$$

$$A_{00} = \frac{\int_0^H\int_0^L f(x,y)\,dx\,dy}{\int_0^H\int_0^L dx\,dy} = \frac{1}{LH}\int_0^H\int_0^L f(x,y)\,dx\,dy$$

$$A_{0m} = \frac{\int_0^H\int_0^L f(x,y)\cos\left(\frac{m\pi y}{H}\right)dx\,dy}{\int_0^H\int_0^L \cos^2\left(\frac{m\pi y}{H}\right)dx\,dy} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{m\pi y}{H}\right)dx\,dy$$

$m = 1, 2, 3, \ldots$

$$A_{n0} = \frac{\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)dx\,dy}{\int_0^H\int_0^L \cos^2\left(\frac{n\pi x}{L}\right)dx\,dy} = \frac{2}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)dx\,dy$$

$n = 1, 2, 3, \ldots$

$$A_{nm} = \frac{\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy}{\int_0^H\int_0^L \cos^2\left(\frac{n\pi x}{L}\right)\cos^2\left(\frac{m\pi y}{H}\right)dx\,dy}$$
$$= \frac{4}{LH}\int_0^H\int_0^L f(x,y)\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)dx\,dy$$

for $n = 1, 2, 3, \ldots$
$m = 1, 2, 3, \ldots$

As $t \to \infty$ $\quad e^{-k\frac{n^2\pi^2}{L^2}t} \to 0$, $e^{-k\frac{m^2\pi^2}{H^2}t} \to 0$
$e^{-k\left(\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{H^2}\right)t} \to 0$

So we have $u(x,y,t) = A_{00} + 0 + 0 + 0$
So as $t \to \infty$ $u(x,y,t) \to A_{00}$
where $A_{00} = \frac{1}{LH}\int_0^H\int_0^L f(x,y)\,dx\,dy$

Homework 8 in PDE   Stefanie Reichart

7.7.11 $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{d\theta^2}\right)$

with $u(a,\theta,t)=0$ $u(r,\theta,0)=0$ $\frac{\partial u}{\partial t}(r,\theta,0)=\alpha(r)\sin 3\theta$

Sep. of Var: $u(r,\theta,t) = \phi(r)g(\theta)h(t)$

lecture $\Rightarrow$ We get the t-equ: $h''+\lambda c^2 h = 0$ with solution

$h(t) = c_1\cos(c\sqrt{\lambda}t) + c_2\sin(c\sqrt{\lambda}t)$ (we will see later that λ must be positive)

and we get the 1st SL-Prob:

$g''+\mu g = 0$ $g(-\pi)=g(\pi)$, $g'(-\pi)=g'(\pi)$

$\Rightarrow \mu_0 = 0$, $\mu_m = m^2$ $g_0(\theta)=1$ $g_m(\theta) = a_m\cos(m\theta)+b_m\sin(m$

and we get the ~~&~~ 2nd SL-Prob

$r\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \lambda^2 r^2\phi - m^2\phi = 0$ with solution:

$\phi_{mn}(r) = c\,J_m(\sqrt{\lambda_{mn}}\,r)$ with $\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$ with z_{mn} zero of mth ord Bessel fcn.

$\Rightarrow u_{mn}(r,\theta,t) = (a_{mn}\cos(m\theta)+b_{mn}\sin(m\theta))J_m(\sqrt{\lambda_{mn}}\,r)\cos(c\sqrt{\lambda_{mn}}\,t) +$
$+ (c_{mn}\cos(m\theta)+d_{mn}\sin(m\theta))J_m(\sqrt{\lambda_{mn}}\,r)\sin(c\sqrt{\lambda_{mn}}\,t)$

$u_{mn}(r,\theta,0) = (a_{mn}\cos(m\theta)+b_{mn}\sin(m\theta))J_m(\sqrt{\lambda_{mn}}\,r) = 0$

$\Rightarrow a_{mn} = b_{mn} = 0$

superposition principle $\Rightarrow$

$$\boxed{u(r,\theta,t) = \sum_{n=1}^{\infty} c_{0n} J_0(\sqrt{\lambda_{0n}}\,r)\sin(c\sqrt{\lambda_{0n}}\,t) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(c_{mn}\cos(m\theta)+d_{mn}\sin(m\theta))J_m(\sqrt{\lambda_{mn}}\,r)\sin(c\sqrt{\lambda_{mn}}\,t)}$$

$\frac{\partial u(r,\theta,t)}{\partial t} = \sum_{n=1}^{\infty} c_{0n}\,c\sqrt{\lambda_{0n}}\,J_0(\sqrt{\lambda_{0n}}\,r)\cos(c\sqrt{\lambda_{0n}}\,t) +$

$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(c_{mn}\,c\sqrt{\lambda_{mn}}\cos(m\theta)+d_{mn}\,c\sqrt{\lambda_{mn}}\sin(m\theta))J_m(\sqrt{\lambda_{mn}}\,r)\cos(c\sqrt{\lambda_{mn}}$

$$\frac{\partial u}{\partial t}(r,\theta,0) = \sum_{n=1}^{\infty} c_{0n} J_0(\sqrt{\lambda_{0n}}\, r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(c_{mn}\, c\sqrt{\lambda_{mn}} \cos(m\theta) + d_{mn}\, c\sqrt{\lambda_{mn}} \sin(m\theta)\right) J_m(\sqrt{\lambda_{mn}}\, r)$$

$$= \alpha(r)\sin(3\theta)$$

$$\Rightarrow \boxed{c_{0n} = 0 \ \forall n \quad c_{mn} = 0 \ \forall m,n \qquad d_{mn} = 0 \ \forall m \neq 3}$$

$$\Rightarrow \frac{\partial u}{\partial t}(r,\theta,0) = \sum_{n=1}^{\infty} d_{3n}\, c\sqrt{\lambda_{3n}} \sin(3\theta) J_3(\sqrt{\lambda_{3n}}\, r) = \alpha(r)\sin(3\theta)$$

with $\boxed{d_{3n} = \dfrac{\int_0^a \alpha(r) J_3(\sqrt{\lambda_{3n}}\, r)\, r\, dr}{c\sqrt{\lambda_{3n}} \int_0^a J_3^2(\lambda_{3n} r)\, r\, dr}}$

and $\boxed{u(r,\theta,t) = \sum_{n=1}^{\infty} d_{3n} \sin(3\theta) J_3(\sqrt{\lambda_{3n}}\, r) \sin(c\sqrt{\lambda_{3n}}\, t)}$

7.91 c $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$

with $u(r,\theta,0) = 0$, $u(r,\theta,H) = \beta(r)\cos(3\theta)$

$\frac{\partial u}{\partial r}(a,\theta,z) = 0$,

$u(r,-\pi,z) = u(r,\pi,z)$, $u_\theta(r,-\pi,z) = u_\theta(r,\pi,z)$

$|u(0,\theta,z)| < \infty$

Separation of Variables $u(r,\theta,z) = \phi(r)g(\theta)h(z)$

$\Rightarrow$ z-dependence: $h'' - \lambda h = 0$

and we get the 1st SL Prob $g'' + \mu g = 0$

with $g(-\pi) = g(\pi)$, $g'(-\pi) = g'(\pi)$

$\Rightarrow \mu_0 = 0$ $g_0(\theta) = 1$ $\mu_m = m^2$ $g_m(\theta) = a_m\cos(m\theta) + b_m\sin(m\theta$

$\Rightarrow$ r-dependence: $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right)\phi = 0$

with $\phi'(a) = 0$ $|\phi(0)| < \infty$

$\Rightarrow \phi(r) = c_1 J_m(\sqrt{\lambda}\, r)$

$\phi'(a) = c_1\sqrt{\lambda}\, J_m'(\sqrt{\lambda}\, a)$

Case 1: $\lambda = 0 \Rightarrow \phi(r) = c_1 J_m(0) = \begin{cases} 1 & m=0 \\ 0 & m\neq 0 \end{cases}$

$g_0(\theta) = a_n$

$\Rightarrow \lambda_{00} = 0$ is an eigenvalue with ef $\phi_{00}(r,\theta) = 1$

$\Rightarrow$ the z-prob becomes $h'' = 0 \Rightarrow h(z) = c_1 z + c_2$

one solution $u_{00}(r,\theta,z) = c_1 z + c_2$

$u_{00}(r,\theta,0) = \underline{c_2 = 0}$

Case 2: $\lambda > 0 \Rightarrow$ let z_{mn} be the nth zero of $J_m'(z)$

$\Rightarrow \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$ $m = 0,1,..$ $n = 1,2..$

$\Rightarrow$ the z-prob becomes $h'' - \lambda_{mn} h = 0$ with $h(0) = 0$

$\Rightarrow h(z) = c_1\cosh(\sqrt{\lambda_{mn}}\, z) + c_2\sinh(\sqrt{\lambda_{mn}}\, z)$

$h(0) = c_1 = 0 \Rightarrow h(z) = c_2\sinh(\sqrt{\lambda_{mn}}\, z)$

$$\Rightarrow u(r,\theta,z) = c_1 z + \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}}\, r)\sinh(\sqrt{\lambda_{0n}}\, z) +$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)) J_m(\sqrt{\lambda_{mn}}\, r)\sinh(\sqrt{\lambda_{mn}}\, z)$$

$$u(r,\theta,H) = c_1 H + \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}}\, r)\sinh(\sqrt{\lambda_{0n}}\, H) +$$

$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (A_{mn}\cos(m\theta) + B_{mn}\sin(m\theta)) J_m(\sqrt{\lambda_{mn}}\, r)\sinh(\sqrt{\lambda_{mn}}\, H) =$$

$$= \beta(r)\cos(3\theta)$$

$$\Rightarrow \int_{-\pi}^{\pi}\int_0^a c_1 H \, dr\, d\theta = \int_{-\pi}^{\pi}\int_0^a \beta(r)\cos(3\theta)\, dr\, d\theta = 0$$

$$\Rightarrow \boxed{c_1 = 0}$$

Let $\boxed{A_{0n} = 0 \ \forall n, \quad B_{mn} = 0 \ \forall m,n \quad A_{mn} = 0 \ \forall m \neq 3, \forall n}$

$$\sum_{n=1}^{\infty} A_{3n}\cos(3\theta) J_3(\sqrt{\lambda_{3n}}\, r)\sinh(\sqrt{\lambda_{3n}}\, H) = \beta(r)\cos(3\theta)$$

$$\Rightarrow \boxed{A_{3n} = \frac{\int_0^a \beta(r) J_3(\sqrt{\lambda_{3n}}\, r)\, r\, dr}{\sinh(\sqrt{\lambda_{3n}}\, H)\int_0^a J_3^2(\sqrt{\lambda_{3n}}\, r)\, r\, dr}}$$