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Math 342B Homework 6

13.2.3 Solve the semi-infinite plate problem if the bottom edge of width π is held at $T = \cos(x)$ and the other sides are at $0°$.

We are solving the two-dimensional Laplace equation

$$\nabla^2 T = 0 \tag{1}$$

for the function $T : [0, \pi] \times [0, \infty) \to \mathbb{R}$ with boundary conditions $T(x, 0) = \cos(x)$ and $T(0, y) = T(\pi, y) = \lim_{y\to\infty} T(x, y) = 0$ for all $y \geq 0$ and $x \in [0, \pi]$. Assume a solution $T(x, y) = X(x)Y(y)$. Then (1) becomes

$$\nabla^2 T = \frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} = Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0 \tag{2}$$

Rearranging terms in (2), we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}. \tag{3}$$

Since we have two functions of independent variables equal to each other, they must be equal to some constant $c \in \mathbb{R}$:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = c. \tag{4}$$

This gives the two second-order ordinary differential equations

$$X'' = cX \qquad \text{and} \qquad Y'' = -cY \tag{5}$$

whose solutions are well-known:

	$X(x)$	$Y(y)$
$c > 0 \implies$	$= Ae^{\sqrt{c}x} + Be^{-\sqrt{c}x}$	$= C\cos(\sqrt{c}y) + D\sin(\sqrt{c}y)$
$c < 0 \implies$	$= A\cos(\sqrt{c}x) + B\sin(\sqrt{c}x)$	$= Ce^{\sqrt{c}y} + De^{-\sqrt{c}y}$

for some $A, B, C, D \in \mathbb{R}$. If we choose $c > 0$ then our Y solution will consist of oscillations. Instead, we want Y to decay to 0 as $y \to \infty$. So, we choose the exponential growth/decay solution for Y, thereby choosing $c < 0$. Let $-k^2 = c$. Then we have the two equations

$$X(x) = A\cos(kx) + B\sin(kx) \qquad \text{and} \qquad Y(y) = Ce^{ky} + De^{-ky} \tag{6}$$

such that

$$T(x, y) = X(x)Y(y) = [A\cos(kx) + B\sin(kx)]\left[Ce^{ky} + De^{-ky}\right]. \tag{7}$$

From the initial condition $T(0, y) = 0$, we have

$$T(0, y) = [A\cos(0) + B\sin(0)]\left[Ce^{ky} + De^{-ky}\right] = A\left[Ce^{ky} + De^{-ky}\right] = 0. \tag{8}$$

Thus, either $A = 0$ or $Ce^{ky} + De^{-ky} = 0$. The functions e^{ky} and e^{-ky} are linearly independent, so the only linear combination of them that gives zero is when the coefficients are zero. Thus, we would have $C = D = 0$. This would give us the trivial solution $T = 0$. We don't want that, so we pick $A = 0$. Therefore, after satisfying the first boundary condition ($T(0, y) = 0$), we are left with

$$T(x, y) = B\sin(kx)\left[Ce^{ky} + De^{-ky}\right]. \tag{9}$$

Next, we look at the boundary condition $\lim_{y\to\infty} T(x,y) = 0$. The Ce^{ky} term of (9) would grow exponentially as $y \to \infty$ if $C \neq 0$, so we must choose $C = 0$ to maintain the condition $T \to 0$ as $y \to \infty$. Thus, letting $K = BD$, our solution in (9) reduces to

$$T(x,y) = K \sin(kx)\mathrm{e}^{-ky}. \tag{10}$$

From the third boundary condition $T(\pi, y) = 0$, we get

$$T(\pi,y) = K \sin(k\pi)\mathrm{e}^{-ky}. \tag{11}$$

Assuming $K \neq 0$ so the solution isn't trivial and since $\mathrm{e}^{-ky} > 0$ for all $k, y \in \mathbb{R}$, it follows that $\sin(k\pi) = 0$. This implies that $k = n$ for some $n \in \mathbb{Z}$. Thus our solution is hardly changed. We define

$$T_n(x,y) = \sin(nx)\mathrm{e}^{-ny}, \qquad n \in \mathbb{N}. \tag{12}$$

By the discussion above, T_n is a solution to $\nabla^2 T = 0$ for each $n \in \mathbb{N}$. Thus, any linear combination of T_n's is a solution as well. Then we have

$$T(x,y) = \sum_{n\in\mathbb{N}} b_n T_n(x,y) = \sum_{n\in\mathbb{N}} b_n \sin(nx)\mathrm{e}^{-ny} \tag{13}$$

with each $b_i \in \mathbb{R}$. We can now solve for the b_n coefficients by looking at the final boundary condition (i.e., $T(x,0) = \cos(x)$). We have

$$T(x,0) = \sum_{n\in\mathbb{N}} b_n \sin(nx) = \cos(x) \tag{14}$$

Finding the coefficients b_n turns into a routine problem of solving for the sine Fourier expansion of $\cos(x)$ on $[0, \pi]$:

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^{\pi} \left[\frac{\mathrm{e}^{ix}+\mathrm{e}^{-ix}}{2}\right]\left[\frac{\mathrm{e}^{inx}-\mathrm{e}^{-inx}}{2i}\right]\mathrm{d}x \\
&= \frac{1}{2i\pi}\int_0^{\pi} \left[\mathrm{e}^{i(n+1)x} - \mathrm{e}^{-i(n-1)x} + \mathrm{e}^{i(n-1)x} - \mathrm{e}^{-i(n+1)x}\right]\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \left[\left(\frac{\mathrm{e}^{i(n+1)x}-\mathrm{e}^{-i(n+1)x}}{2i}\right) + \left(\frac{\mathrm{e}^{i(n-1)x}-\mathrm{e}^{-i(n-1)x}}{2i}\right)\right]\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^{\pi} \left[\sin([n+1]x) + \sin([n-1]x)\right]\mathrm{d}x \\
&= -\frac{1}{\pi}\left[\frac{1}{n+1}\cos([n+1]x) + \frac{1}{n-1}\cos([n-1]x)\right]_0^{\pi} \\
&= -\frac{1}{\pi}\left[\frac{1}{n+1}\cos([n+1]\pi) + \frac{1}{n-1}\cos([n-1]\pi) - \frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= -\frac{1}{\pi}\left[\frac{2n}{n^2-1}\cos([n+1]\pi) - \frac{2n}{n^2-1}\right] \\
&= -\frac{2n}{\pi(n^2-1)}\left[\cos([n+1]\pi) - 1\right] \\
&= \begin{cases} \dfrac{4n}{\pi(n^2-1)}, & \text{if } n \equiv 0 \pmod{2}, \\ 0, & \text{if } n \equiv 1 \pmod{2}. \end{cases}
\end{aligned} \tag{15}$$

Thus, we can finally write the general solution to $\nabla^2 T = 0$. We have

$$T(x, y) = \frac{8}{\pi} \sum_{n \in \mathbb{N}} \frac{n}{4n^2 - 1} \sin(2nx) e^{-2ny}. \tag{16}$$

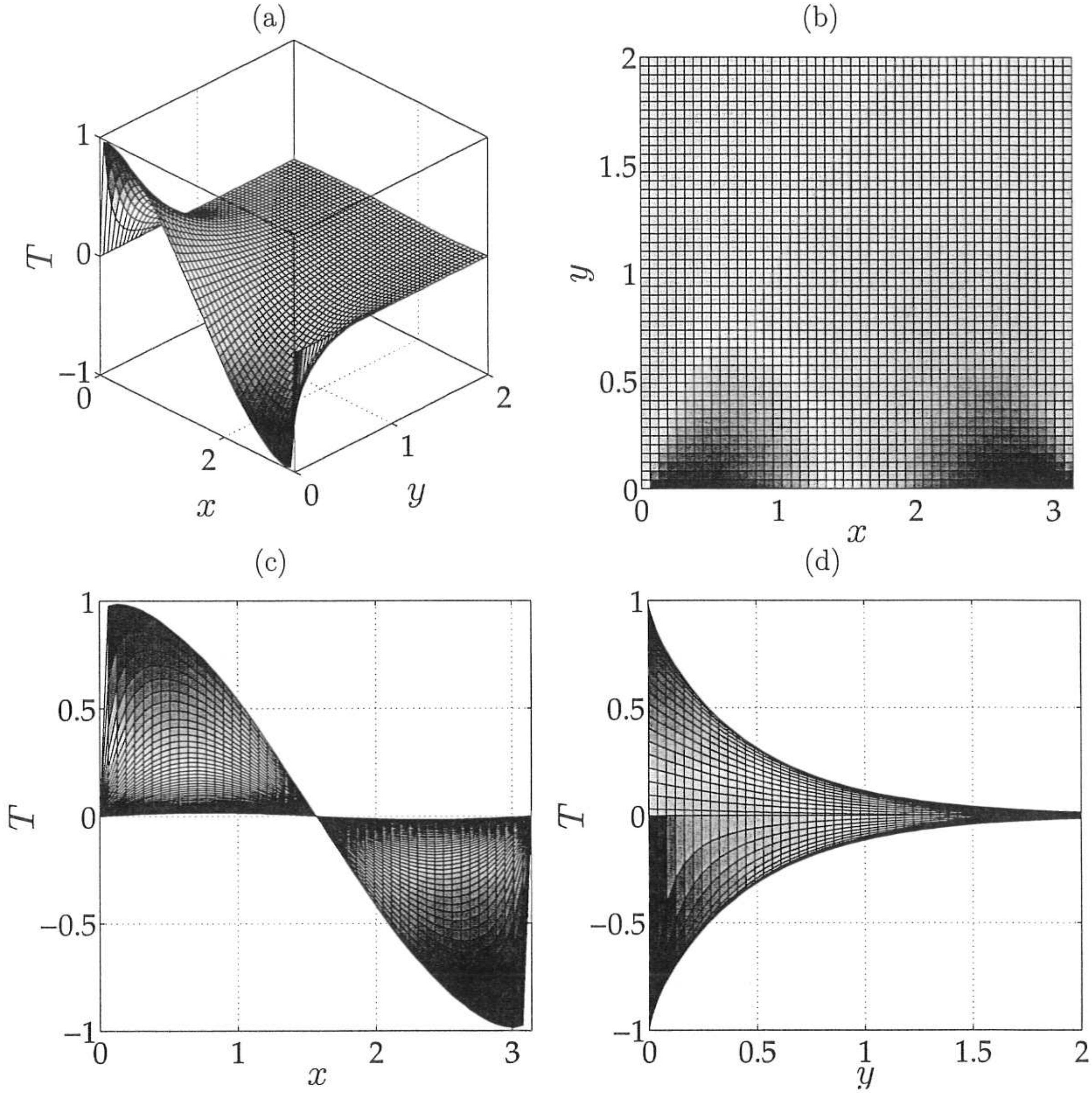


FIGURE 1. Partial sum up to $n = 200$ of the solution (16) of $\nabla^2 T = 0$ on the semi-infinite region $[0, \pi] \times [0, \infty)$ with the following boundary conditions:

(1) $T(0, y) = 0$,
(2) $T(\pi, y) = 0$,
(3) $\lim_{y \to \infty} T(x, y) = 0$,
(4) $T(x, 0) = \cos(x)$.

Plots show the region $[0, \pi] \times [0, 2] \times [-1, 1]$.

13.2.10 Find the steady-state temperature distribution in a metal plate 10 cm square if one side is held at 100° and the other three at 0°. Find the temperature at the center of the plate.

We are looking for the function $T : [0,10]\times[0,10] \to \mathbb{R}$ such that $\nabla^2 T = 0$, $T(x,0) = 100$, and $T(0,y) = T(10,y) = T(x,10) = 0$. We assume a solution $T(x,y) = X(x)Y(y)$. Following the same initial steps as in the last problem (equations (1)-(4)), we again obtain

$$X'' = cX \qquad \text{and} \qquad Y'' = -cY. \tag{17}$$

Choosing $c = -k^2$ as before, we get

$$X(x) = A\cos(kx) + B\sin(kx) \qquad \text{and} \qquad Y(y) = Ce^{ky} + De^{-ky} \tag{18}$$

for $A, B, C, D \in \mathbb{R}$ so that

$$T(x,y) = X(x)Y(y) = [A\cos(kx) + B\sin(kx)]\big[Ce^{ky} + De^{-ky}\big]. \tag{19}$$

From the boundary condition $T(0,y) = 0$, it follows that $A = 0$ for the same reason as in the last problem. Thus,

$$T(x,y) = B\sin(kx)\big[Ce^{ky} + De^{-ky}\big]. \tag{20}$$

Our solution starts to differ from the solution of the previous problem when we consider the (no longer infinite) boundary condition $T(x,10) = 0$. Then we have

$$T(x,10) = B\sin(kx)\big[Ce^{10k} + De^{-10k}\big] = 0, \tag{21}$$

so we must have $Ce^{10k} + De^{-10k} = 0$ in order to not have a trivial solution. Note that if we choose $C = \frac{1}{2}e^{-10k}$ and $D = \frac{1}{2}e^{10k}$, then we have

$$Ce^{ky} + De^{-ky} = \frac{1}{2}e^{-(10-y)k} + \frac{1}{2}e^{(10-y)k} = \sinh(k[10-y]) \tag{22}$$

and so

$$T(x,10) = B\sin(kx)\sinh(0) = 0, \tag{23}$$

as desired. Thus our solution reduces to

$$T(x,y) = B\sin(kx)\sinh(k[10-y]).$$

Furthermore, from the boundary condition $T(10,y) = 0$, we have

$$T(10,y) = B\sin(k10)\sinh(k[10-y]) = 0, \tag{24}$$

so we must have $k = \frac{n\pi}{10}$ for some $n \in \mathbb{N}$. Now define

$$T_n(x,y) = \sin\left(\frac{n\pi}{10}x\right)\sinh\left(\frac{n\pi}{10}[10-y]\right) \tag{25}$$

so that T_n solves $\nabla^2 T = 0$. Then an arbitrary linear combination of T_n's solves $\nabla^2 T = 0$. Thus, we have

$$T(x,y) = \sum_{n\in\mathbb{N}} b_n T_n(x,y) = \sum_{n\in\mathbb{N}} b_n \sin\left(\frac{n\pi}{10}x\right)\sinh\left(\frac{n\pi}{10}[10-y]\right). \tag{26}$$

From the last boundary condition $T(x,0) = 100$, we have

$$T(x,0) = \sum_{n\in\mathbb{N}} b_n \sin\left(\frac{n\pi}{10}x\right)\sinh(n\pi) = 100 \implies \sum_{n\in\mathbb{N}} b_n \sin\left(\frac{n\pi}{10}x\right) = \frac{100}{\sinh(n\pi)}. \tag{27}$$

Thus, the b_n coefficients are the sine Fourier coefficients of the constant function $\frac{100}{\sinh(n\pi)}$ on the interval $[0, 10]$. So, we have

$$\begin{aligned}
b_n &= \frac{1}{5}\int_0^{10} \frac{100}{\sinh(n\pi)} \sin\left(\frac{n\pi}{10}x\right) dx \\
&= \frac{20}{\sinh(n\pi)} \int_0^{10} \sin\left(\frac{n\pi}{10}x\right) dx \\
&= -\frac{200}{n\pi\sinh(n\pi)} \left[\cos\left(\frac{n\pi}{10}x\right)\right]_0^{10} \\
&= -\frac{200}{n\pi\sinh(n\pi)} [\cos(n\pi) - 1] \\
&= \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ \dfrac{400}{n\pi\sinh(n\pi)}, & \text{if } n \equiv 1 \pmod 2. \end{cases}
\end{aligned} \tag{28}$$

So, our solution is

$$\boxed{T(x,y) = \frac{400}{\pi} \sum_{n\in\mathbb{N}\text{ odd}} \frac{1}{n\sinh(n\pi)} \sin\left(\frac{n\pi}{10}x\right) \sinh\left(\frac{n\pi}{10}[10-y]\right).} \tag{29}$$

Moreover, the center of the plate is the point $(5, 5)$. MATLAB computed $T(5, 5)$ up to $n = 200$. According to MATLAB (and coinciding with the solution in the textbook),

$$\boxed{T(5,5) = 25^\circ.} \tag{30}$$

HW #6

1) a) $u_t = ku_{xx}$

$u(0,t) = 0 \quad u(10,t) = 100 \qquad 0 < x < 10 \;, t > 0$

For steady-state temperature distribution, we have $u(x,t) = u(x)$ and thus $u_t = 0$

$0 = u_{xx}$

$\Rightarrow u(x) = c_1 x + c_2$

$u(0) = c_2 \overset{!}{=} 0 \quad \Rightarrow c_2 = 0$

$u(10) = c_1 \cdot 10 \overset{!}{=} 100 \quad \Rightarrow c_1 = 10$

$\Rightarrow u_E(x) = 10x$

b) $u_t = u_{xx}$

BC: $u_x(0,t) = u_x(L,t) = 0$

IC: $u(x,0) = u_E(x)$

Let $u(x,t) = \phi(x)G(t)$

$G'\phi = \phi'' G$

$\frac{1}{G}G' = \frac{1}{\phi}\phi'' = -\lambda$

- Solution of $G' = -\lambda G$ is $G(t) = ce^{-\lambda t}$
- Solution of SL-Problem $\phi'' + \lambda\phi = 0 \;, \; \phi'(0) = \phi'(L) = 0$

$\underline{\lambda = 0}$: $\phi(x) = c_1 x + c_2$

$\phi'(0) = c_1 \overset{!}{=} 0 \quad \Rightarrow c_1 = 0$

$\phi'(L) = c_1 = 0$

$\Rightarrow \phi(x) = c_2$ arbitrary

$\lambda_0 = 0$ eval with efct $\phi_0(x) = 1$

$\underline{\lambda<0}$. $\phi'' + \lambda\phi = 0$ has char. poly $r^2+\lambda = 0$ $\Rightarrow r = \pm\sqrt{-\lambda}$

$\Rightarrow \phi(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$\phi'(x) = \sqrt{-\lambda}\left[c_1 e^{\sqrt{-\lambda}x} - c_2 e^{-\sqrt{-\lambda}x}\right]$

BC:

$\phi'(0) = \sqrt{-\lambda}\,[c_1 - c_2] \overset{!}{=} 0 \Rightarrow c_1 = c_2$

$\phi'(L) = \sqrt{-\lambda}\,c_1\left[e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right] \overset{!}{=} 0$

Since $e^{\sqrt{-\lambda}L} \neq e^{-\sqrt{-\lambda}L}$ $\forall L \neq 0$, c_1 must be 0

Hence only trivial solution and no neg. evals

$\underline{\lambda>0}$: char. poly $r^2+\lambda=0$ $\Rightarrow r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$

$\Rightarrow \phi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$\phi'(x) = \sqrt{\lambda}\left[-c_1\sin\sqrt{\lambda}x + c_2\cos\sqrt{\lambda}x\right]$

$\phi'(0) = \sqrt{\lambda}\,c_2 \overset{!}{=} 0 \Rightarrow c_2 = 0$

$\phi'(L) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}L \overset{!}{=} 0$

We are looking for nontrivial solutions, hence

$\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2, n\geq 1$

$\phi_n(x) = \cos\frac{n\pi x}{L}, n\geq 1$

With $\lambda_0 = 0$, $\phi_0(x) = 1$ all valid for $n\geq 0$.

$u_n(x,t) = A_n\cos\frac{n\pi x}{L}\,e^{-\left(\frac{n\pi}{L}\right)^2 t}$

Superposition:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\,e^{-\frac{n^2\pi^2}{L^2}t}$$

$$= \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{L}\,e^{-\frac{n^2\pi^2}{L^2}t}$$

IC.

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{L}$$

Multiplying by $\cos\frac{m\pi x}{L}$ and integrating

$$\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx = \sum_{n=0}^{\infty} A_n \int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx$$

where $$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ L/2 & \text{if } n = m \neq 0 \\ L & \text{if } n = m = 0 \end{cases}$$

<u>$n=m=0$:</u>

$$\int_0^L f(x) = L A_0 \Rightarrow A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^L 10x\,dx$$

$$= \frac{1}{L}\cdot 5L^2 = 5L = 50$$

<u>$n=m\neq 0$:</u>

$$\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{L}{2}A_n$$

$$\Rightarrow A_n = \frac{2}{L}\int_0^L 10x\cos\frac{n\pi x}{L}\,dx = \frac{20}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{20}{L}\left\{ x\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L - \int_0^L \frac{L}{n\pi}\sin\frac{n\pi x}{L}\,dx \right\}$$

$$= \frac{20}{L}\left[\frac{xL}{n\pi}\sin\frac{n\pi x}{L} - \frac{L}{n\pi}\cdot\frac{L}{n\pi}\left(-\cos\frac{n\pi x}{L}\right)\right|_0^L$$

$$= \frac{20}{L}\left[x\frac{L}{n\pi}\sin\frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2\cos\frac{n\pi x}{L}\right|_0^L$$

$$= \frac{20}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos n\pi - \left(\frac{L}{n\pi}\right)^2\right]$$

$$= \frac{20L}{n^2\pi^2}\left[\cos n\pi - 1\right] = \frac{200}{n^2\pi^2}\left[(-1)^n - 1\right]$$

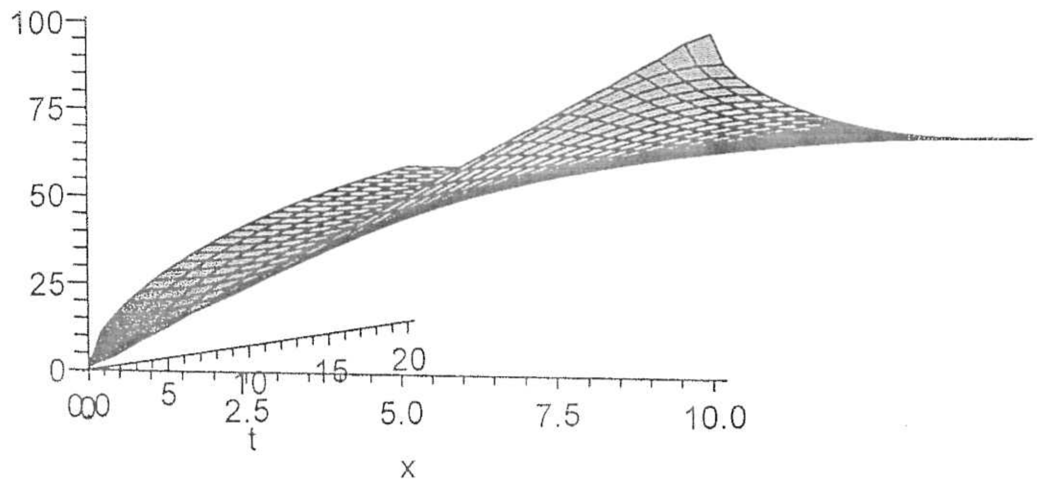
Thus

$$u(x,t) = 50 + \frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi x}{10}\, e^{-\frac{n^2\pi^2}{100}t}$$

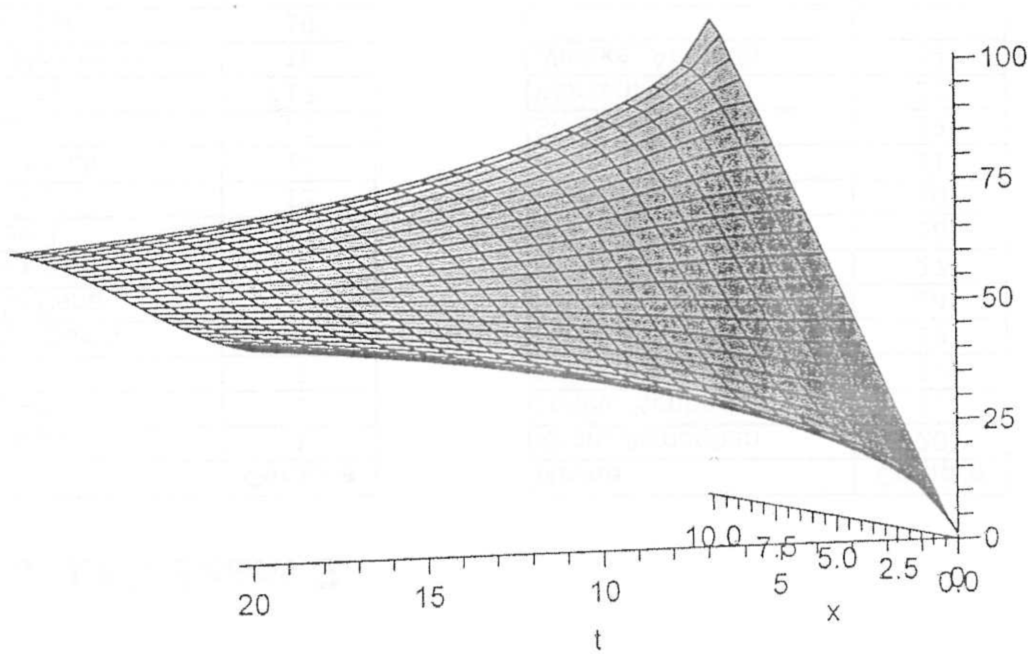
> $u := (x, t) \rightarrow 50 + \left(200/\pi^2\right) * sum\left(\left(\left((-1)^{\wedge}n - 1\right)/n^2\right) * \cos(n * \mathrm{Pi} * x/10) * \exp(-(n * \mathrm{Pi}/10)^{\wedge}2 * t), n = 1..20\right);$

$$u := (x, t) \rightarrow 50 + \frac{200\left(\sum_{n=1}^{20} \frac{\left((-1)^n - 1\right)\cos\left(\frac{1}{10}\, n\,\pi\, x\right) \mathrm{e}^{-\frac{1}{100} n^2 \pi^2 t}}{n^2}\right)}{\pi^2} \tag{1}$$

> $plot3d(u(x, t), x = 0..10, t = 0..20);$



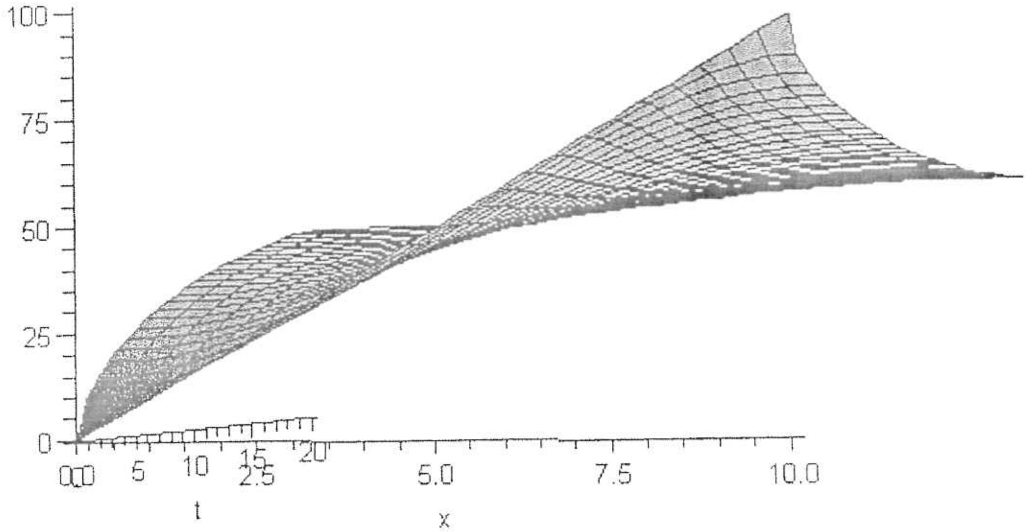
> $plot3d(u(x, t), x = 0..10, t = 0..20);$



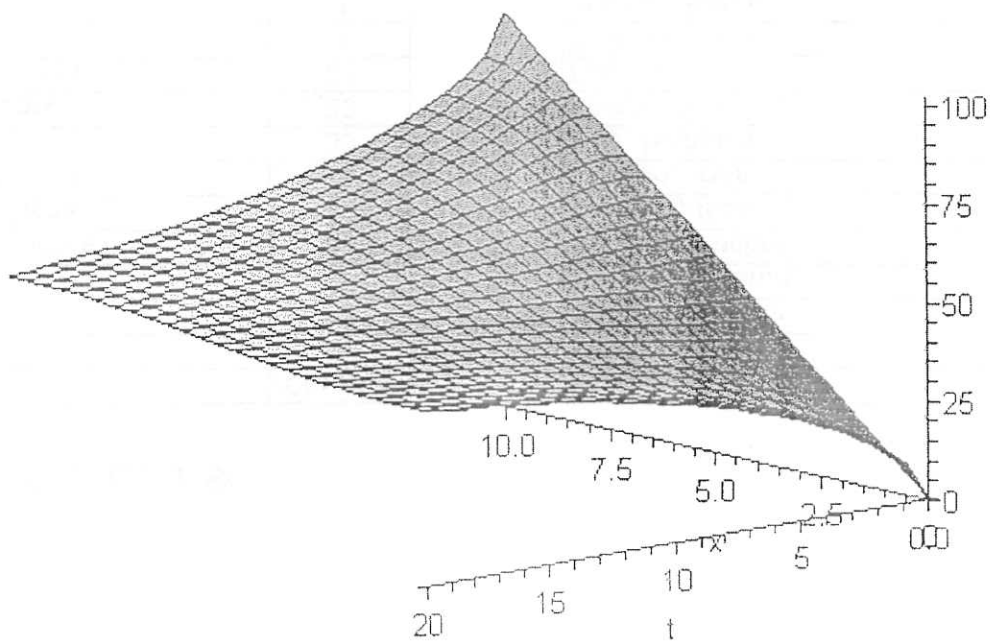
> $u := (x, t) \rightarrow 50 + \left(200/\pi^2\right) * sum\left(\left(\left((-1)\text{^}n - 1\right)/n^2\right) * \cos(n * \mathrm{Pi} * x/10) * \exp(-(n * \mathrm{Pi}/10)\text{^}2 * t), n = 1..200\right);$

$$u := (x, t) \rightarrow 50 + \frac{200 \left(\sum_{n=1}^{200} \frac{\left((-1)^n - 1\right) \cos\left(\frac{1}{10}\, n\, \pi\, x\right) \mathrm{e}^{-\frac{1}{100} n^2 \pi^2 t}}{n^2} \right)}{\pi^2} \tag{1}$$

> $plot3d(u(x, t), x = 0..10, t = 0..20);$



> $plot3d(u(x, t), x = 0..10, t = 0..20);$



(2) $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = 0, \quad u(x,0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}, \quad \frac{\partial u}{\partial y}(x,H) = 0$

Assume: $u(x,y) = \Phi(x) \cdot G(y)$ ⇒ similar to part a) we get the two ODE

$\Phi''(x) + \lambda \Phi(x) = 0, \quad \Phi'(0) = 0 \quad \Phi'(L) = 0$ (SL-Problem)

$G''(y) - \lambda G(y) = 0 \qquad G'(H) = 0$

For the SL-Problem we get the EVs: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ and the eigenfunctions: $\Phi_n = \cos\left(\frac{n\pi x}{L}\right) \quad n = 0, 1, 2 \ldots$

For the second problem, we get the solution

$G(y) = c_1 \cosh\left(\frac{n\pi}{L}(H-y)\right) + c_2 \sinh\left(\frac{n\pi}{L}(H-y)\right)$

$\Rightarrow G'(y) = -c_1 \frac{n\pi}{L} \sinh\left(\frac{n\pi}{L}(H-y)\right) - c_2 \cosh\left(\frac{n\pi}{L}(H-y)\right) \cdot \frac{n\pi}{L}$

$\Rightarrow G'(H) = -c_2 \frac{n\pi}{L} = 0 \Rightarrow c_2 = 0$

$\Rightarrow G(y) = c_1 \cosh\left(\frac{n\pi}{L}(H-y)\right)$

⇒ general solution: $u(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi}{L}(H-y)\right) =$

$= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi}{L}(H-y)\right)$

$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi H}{L}\right) = f(x) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}$

$\Rightarrow A_0 = \frac{1}{L} \int_0^L f(x)\,dx = \frac{1}{L} \int_0^{L/2} 1\,dx = \frac{1}{2}$

$\int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi H}{L}\right) \int_0^L \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$

$\Rightarrow A_n = \frac{2}{L \cosh\left(\frac{n\pi H}{L}\right)} \int_0^{L/2} \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{\cancel{L} \cosh\left(\frac{n\pi H}{L}\right)} \frac{\cancel{L}}{n\pi} \sin\left(\frac{n\pi \cancel{L}}{2\cancel{L}}\right) =$

$= \frac{2 \sin\left(\frac{n\pi}{2}\right)}{n\pi \cosh\left(\frac{n\pi H}{L}\right)}$

$$\Rightarrow \boxed{u(x,y) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2 \sin\left(\frac{n\pi}{2}\right)}{n\pi \cosh\left(\frac{n\pi H}{L}\right)} \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi}{L}(H-y)\right)}$$

③ $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad 0<r<a, \quad 0<\theta<\pi$

a) $u(r,0)=0 \quad 0\le r\le a, \quad u(r,\pi)=0 \quad 0\le r\le a$

$u(a,\theta)=f(\theta) \qquad 0<\theta<\pi$

Assume $u(r,\theta) = \cancel{\quad} \phi(\theta)G(r)$

$\Rightarrow \frac{\phi}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right) + \frac{1}{r^2}\phi'' G = 0$

$\Leftrightarrow -\frac{\phi''}{\phi} = \frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda$

We get the two ODEs:

$\phi'' + \lambda\phi = 0 \qquad \phi(0)=0, \quad \phi(\pi)=0 \qquad$ (SL-Prob)

$rG'' + G' - \frac{1}{r}\lambda G = 0$

For the SL-Prob we get the EVs $\lambda_n = n^2$ and the eigenf

$\phi_n = \sin(n\theta), \quad n=1,2,\dots$

$\Rightarrow$ The second ODE becomes $rG'' + G' - \frac{n^2}{r}G = 0$

see lecture, $u(0,0)=0$ $\Rightarrow$ $G_n(r) = cr^n$

$\Rightarrow$ general solution: $u(r,\theta) = \sum_{n=1}^{\infty} a_n r^n \sin(n\theta)$

$\Rightarrow u(a,\theta) = \sum_{n=1}^{\infty} a_n a^n \sin(n\theta) = g(\theta)$

$\Rightarrow \int_0^{\pi} g(\theta)\cancel{d\theta}\sin(m\theta)\,d\theta = \sum_{n=1}^{\infty} a_n a^n \underbrace{\int_0^{\pi}\sin(n\theta)\sin(m\theta)\,d\theta}_{\substack{\pi/2 \;\; m=n \\ 0 \;\; \text{else}}}$

$\Rightarrow a_n = \frac{2}{\pi a^n}\int_0^{\pi} g(\theta)\sin(n\theta)\,d\theta$

$\Rightarrow u(\cancel{\theta}r,\theta) = \sum_{n=1}^{\infty}\frac{2}{\pi a^n}\int_0^{\pi} g(\theta)\sin(n\theta)\,d\theta \cdot r^n\sin(n\theta) =$

$$\boxed{= \sum_{n=1}^{\infty}\frac{2}{\pi}\int_0^{\pi} g(\theta)\sin(n\theta)\,d\theta\left(\frac{r}{a}\right)^n\sin(n\theta)}$$

b) $\frac{\partial u}{\partial r}(r,0)=0, \quad \frac{\partial u}{\partial r}(r,\pi)=0, \quad u(a,\theta)=f(\theta)$

like in part a) we get the two ODEs:

$\phi'' + \lambda\phi = 0, \quad \phi'(0)=0, \quad \phi'(\pi)=0 \qquad$ (SL-Prob)

$rG'' + G' - \frac{1}{r}\lambda G = 0$

solution of SL Prob: EVs: $\lambda_n = n^2$ eig-fcns: $\phi_n = \cos(n\theta)$

$G_n(r) = c r^n$ $\qquad \underline{n = 0 \ !}$

$\Rightarrow$ general solution: $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cancel{a} r^n \cos(n\theta)$

$\Rightarrow u(a,\theta) = a_0 + \sum_{n=1}^{\infty} a_n a^n \cos(n\theta) = g(\theta)$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{\pi} g(\theta)\, d\theta$

$$\int_0^{\pi} g(\theta)\cos(m\theta)\, d\theta = \sum_{n=1}^{\infty} a_n a^n \int_0^{\pi} \overbrace{\cos(n\pi)\cos(m\pi)}^{\substack{\pi/2 \ \ m=n \\ 0 \ \ \text{else}}}\, d\pi$$

$\Rightarrow a_n a^n = \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos(n\theta)\, d\theta \Rightarrow a_n = \frac{2}{\pi a^n}\int_0^{\pi} g(\theta)\cos(n\theta)\, d\theta$

$$\Rightarrow \boxed{u(r,\theta) = \frac{1}{\pi}\int_0^{\pi} g(\theta)\, d\theta + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{2}{\pi}\int_0^{\pi} g(\theta)\cos(n\theta)\, d\theta \cdot \cos(n\theta)}$$

(4) $$\frac{\partial^2 u}{\partial t^2} + 0.2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0<x<1, \quad t>0$$

B.C. $u(0,t)=0, \quad u(1,t)=0$

I.C. $u(x,0)=0, \quad u_t(x,0)=1$

$u(x,t)=X(x)\,T(t) \qquad X T'' + 0.2 X T' = X'' T$

$$\frac{T''+0.2T'}{T} = \frac{X''}{X} = -\lambda$$

SL Prob $\quad X''+\lambda X=0 \qquad X(0)=0, \ X(1)=0$

Shown before $\quad \lambda = n^2\pi^2$ e.v. with $X_n(x)=\sin(n\pi x)$

T-eqn $\quad T_n'' + 0.2T_n' + n^2\pi^2 T_n = 0 \qquad$ Char Eqn. $r^2+0.2r+n^2\pi^2=0$

$r=-0.1 \pm i\sqrt{n^2\pi^2-0.01} \equiv -0.1 \pm i\omega_n$ with $\omega_n=\sqrt{n^2\pi^2-0.01}$

$T_n(t)=e^{-0.1t}(a_n\cos(\omega_n t)+b_n\sin(\omega_n t))$

$$u(x,t)=\sum_{n=1}^{\infty} e^{-0.1t}(a_n\cos(\omega_n t)+b_n\sin(\omega_n t))\sin(n\pi x)$$

$u(x,0)=0 \Rightarrow a_n=0$

$$u_t(x,t)=\sum_{n=1}^{\infty} e^{-0.1t}\, b_n(\omega_n\cos(\omega_n t)-0.1\sin(\omega_n t))\sin(n\pi x)$$

$$u_t(x,0)=\sum_{n=1}^{\infty} b_n\omega_n\sin(n\pi x)=1$$

$$b_n\omega_n = \frac{\int_0^1 \sin(n\pi x)\cdot 1\,dx}{\int_0^1 \sin^2(n\pi x)\,dx} = \frac{2(1-\cos(n\pi))}{n\pi}$$

$$b_n=\frac{2}{n\pi\omega_n}(1-(-1)^n)$$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi\omega_n}e^{-0.1t}\sin(\omega_n t)\sin(n\pi x)$$

$\lim_{t\to 0} u(x,t)=0$, since exponentially decaying

⑤ $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0$, B.C. $u(1,\theta)=\begin{cases}0, & 0\le\theta\le\pi/2\\ 1, & \pi/2<\theta\le\pi\end{cases}$, $u(2,\theta)=\begin{cases}2 & 0\le\theta\le\pi/2\\ 0 & \pi/2<\theta\le\pi\end{cases}$

$u_\theta(r,0)=0$, $u_\theta(r,\pi)=0$. Let $u(r,\theta)=h(r)\phi(\theta)$. $\frac{\phi}{r}(rh')'+\frac{h\phi''}{r^2}=0$

$\Rightarrow \frac{r(rh')'}{h}=-\frac{\phi''}{\phi}=\lambda$ SL Prob $\phi''+\lambda\phi=0$, $\phi'(0)=0$ and $\phi'(\pi)=0$.

Have shown e.v. $\lambda_0=0$, e.f. $\phi_0(\theta)=1$ and e.v. $\lambda_n=n^2$, e.f. $\phi_n(\theta)=\cos(n\theta)$.

The r-eqn. becomes $r(rh_n')'=n^2h$ or $r^2h''+rh'-n^2h=0$. For $\lambda_0=0$;

$(rh_0')'=0\Rightarrow h_0(r)=a_0\ln(r)+b_0$. If $\lambda_n=n^2$, $n=1,2,\dots$, then $h_n(r)=a_nr^{-n}+b_nr^n$.

Superposition gives $u(r,\theta)=a_0\ln(r)+b_0+\sum_{n=1}^{\infty}(a_nr^{-n}+b_nr^n)\cos(n\theta)$.

From B.C's $u(1,\theta)=b_0+\sum_{n=1}^{\infty}(a_n+b_n)\cos(n\theta)=\left\{\begin{matrix}0, & 0\le\theta\le\pi/2\\ 1 & \pi/2<\theta\le\pi\end{matrix}\right\}=f(\theta)$

$u(2,\theta)=a_0\ln(2)+b_0+\sum_{n=1}^{\infty}(a_n(2)^{-n}+b_n(2)^n)\cos(n\theta)=\left\{\begin{matrix}2, & 0\le\theta\le\pi/2\\ 0 & \pi/2<\theta\le\pi\end{matrix}\right\}=g(\theta)$

Mult. by $\phi_0(\theta)=1$ & integrate $\theta=0$ to π. $b_0\int_0^\pi d\theta=\int_0^\pi f(\theta)\,d\theta=\int_{\pi/2}^{\pi}1\,d\theta\Rightarrow b_0=\frac{1}{2}$

$(a_0\ln(2)+b_0)\int_0^\pi d\theta=\int_0^\pi g(\theta)d\theta=\int_0^{\pi/2}2\,d\theta\Rightarrow a_0\ln(2)+b_0=1\Rightarrow a_0=\frac{1}{2\ln(2)}$

Mult. by $\phi_k(\theta)=\cos(k\theta)$ & integrate $\theta=0$ to π. $(a_k+b_k)\int_0^\pi\cos^2(k\theta)d\theta=\int_0^\pi f(\theta)\cos(k\theta)d\theta=\int_{\pi/2}^{\pi}\cos(k\theta)\,d\theta$

$\therefore a_k+b_k=-\frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right)$. Similarly, $[a_k(2)^{-k}+b_k(2)^k]\frac{\pi}{2}=\int_0^\pi g(\theta)\cos(k\theta)d\theta=2\int_0^{\pi/2}\cos(k\theta)d\theta\Rightarrow$

$a_k(2)^{-k}+b_k(2)^k=\frac{4}{k\pi}\sin\left(\frac{k\pi}{2}\right)$. Solving for a_k & b_k, $a_k=-\frac{(2^{k+1}+4)\sin\left(\frac{k\pi}{2}\right)}{\pi k(2^k-2^{-k})}$ and

$b_k=\frac{(4+2^{1-k})\sin\left(\frac{k\pi}{2}\right)}{\pi k(2^k-2^{-k})}$. It follows that the solution is

$$u(r,\theta)=\frac{\ln(r)}{2\ln(2)}+\frac{1}{2}+\sum_{k=1}^{\infty}\frac{\sin\left(\frac{k\pi}{2}\right)}{\pi k(2^k-2^{-k})}\left(-(2^{k+1}+4)r^{-k}+(4+2^{1-k})r^k\right)\cos(k\theta)$$

⑥ $u_t = k\,u_{xx}$ $\quad 0 < x < 2,\ t > 0$

B.C. $\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(2,t) = -0.5\,u(2,t)$

I.C. $u(x,0) = f(x)$

a. $u(x,t) = X(x)T(t)$ $\qquad XT' = kX''T$

$\frac{T'}{kT} = \frac{X''}{X} = -\lambda$

S-L Prob $\quad X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(2) = -0.5X(2)$

Trivial soln. for $\lambda \le 0$, $\quad$ Let $\lambda = \alpha^2 > 0$

$X(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x), \quad X'(x) = -c_1\alpha\sin(\alpha x) + c_2\alpha\cos(\alpha x)$

$X'(0) = 0 \Rightarrow c_2 = 0$

$X'(2) = -0.5X(2) \Rightarrow -c_1\alpha\sin(2\alpha) = -0.5c_1\cos(2\alpha)$

Nontrivial soln. when $\tan(2\alpha) = \frac{1}{2\alpha}$

$\alpha_1 = 0.43017, \quad \alpha_2 = 1.7128, \quad \alpha_3 = 3.2186, \quad \alpha_4 = 4.7647, \quad \alpha_5 = 6.3226$

$\therefore\ \lambda_1 = 0.18504, \quad \lambda_2 = 2.9337, \quad \lambda_3 = 10.3597, \quad \lambda_4 = 22.7021, \quad \lambda_5 = 39.9758$

$X_n(x) = \cos(\alpha_n x)$

b. Orthogonality: show $\int_0^2 \cos(\alpha_n x)\cos(\alpha_m x)\,dx = 0$.

pf $\quad$ consider $\lambda_n \ne \lambda_m$ with $X_n'' = -\lambda_n X_n$ & $X_m'' = -\lambda_m X_m$

$X_n'' X_m - X_m'' X_n = (\lambda_m - \lambda_n)X_n X_m$

$$\int_0^1 (X_n'' X_m - X_m'' X_n)\,dx \overset{\text{parts}}{=} X_n' X_m - X_m' X_n \Big|_0^2 - \int_0^2 (X_n' X_m' - X_m' X_n')\,dx$$

$$= \left(X_n'(2)X_m(2) - X_m'(2)X_n(2)\right) - \left(\cancelto{0}{X_n'(0)}X_m(0) - \cancelto{0}{X_m'(0)}X_n(0)\right)$$

$$= -0.5X_n(2)X_m(2) + 0.5X_m(2)X_n(2) = 0 = (\lambda_m - \lambda_n)\int_0^2 X_n(x)X_m(x)\,dx$$

Thus, $\int_0^2 X_n(x)X_m(x)\,dx = 0$ for $\lambda_m \ne \lambda_n$

c. $T_n(t) = e^{-k\lambda_n t}$

$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t}\cos(\alpha_n x)$, $\quad \lambda_n$ & α_n defined above

$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n\cos(\alpha_n x), \quad a_n = \dfrac{\int_0^2 f(x)\cos(\alpha_n x)\,dx}{\int_0^2 \cos^2(\alpha_n x)\,dx}$

Note: this doesn't $= 1$