Sp 2012 — Take Home Exam 2 — Math 342B

① Let $u(x,t) = \phi(x)\,h(t)$ $\qquad \dfrac{h'}{0.2h} = \dfrac{\phi''}{\phi} = -\lambda$

Time Eqn $h' + 0.2\lambda h = 0$

$h(t) = c\,e^{-0.2\lambda t}$

SL Prob $\phi'' + \lambda\phi = 0$, $\quad \phi'(0) = 0$, $\phi'(4) = 0$

Have shown $\lambda < 0 \Rightarrow$ trivial

e.v. $\lambda_0 = 0$ with e.f $\phi_0(x) = 1$

$\lambda = \alpha^2 > 0$, $\quad \lambda_n = \dfrac{n^2\pi^2}{16}$ (e.v.) with $\phi_n(x) = \cos\left(\dfrac{n\pi x}{4}\right)$

superposition

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{0.2 n^2\pi^2 t}{16}} \cos\left(\frac{n\pi x}{4}\right)$$

$$u(x,0) = 5 - 3\cos(\pi x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{4}\right)$$

By orthogonality $A_0 = 5$, $A_4 = -3$, $A_n = 0$ for $n \neq 0, 4$

$$\lim_{t\to\infty} u(x,t) = A_0 = 5$$

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2. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 0.5\frac{\partial u}{\partial x}$, $t>0$, $0<x<10$, $u(0,t)=0$, $u(10,t)=0$, $u(x,0)=f(x)$

Let $u(x,t) = \phi(x)h(t) \Rightarrow \phi h' = \phi'' h - 0.5\phi' h \Rightarrow \frac{h'}{h} = \frac{\phi'' - 0.5\phi'}{\phi} = -\lambda$

a. t-eqn. $h' + \lambda h = 0$; so $h(t) = ce^{-\lambda t}$

SL-Prob. $\phi'' - 0.5\phi' + \lambda\phi = 0$, $\phi(0)=0$, $\phi(10)=0$

b. SL form $[p\phi']' + q\phi + \lambda\sigma\phi = 0$, $H\phi'' - 0.5H\phi' + \lambda H\phi = 0$

$p\phi'' + p'\phi' + q\phi + \lambda\sigma\phi = 0 \Rightarrow H=p \Rightarrow p' = H' = -0.5H \Rightarrow H(x) = e^{-0.5x} = p(x)$

SL form $[e^{-0.5x}\phi']' + \lambda e^{-0.5x}\phi = 0$. $\therefore$ weighting factor $\sigma(x) = e^{-0.5x}$, $q(x)=0$

Char. poly $r^2 - 0.5r + \lambda = (r-1/4)^2 + \lambda - 1/16 = 0$. If $\lambda - 1/16 = -\alpha^2 < 0$, then $r = \frac{1}{4} \pm \alpha$

$\phi(x) = e^{x/4}(c_1\cosh(\alpha x) + c_2\sinh(\alpha x))$ $\phi(0) = c_1 = 0$, $\phi(10) = c_2 e^{10/4}\sinh(10\alpha) = 0 \Rightarrow c_2 = 0$

trivial soln. If $\lambda = 1/16$, $\phi(x) = e^{x/4}(c_1 + c_2 x)$. $\phi(0) = c_1 = 0$, $\phi(10) = e^{10/4}\cdot c_2 10 = 0$

$\Rightarrow c_2 = 0$ trivial soln. If $\lambda - 1/16 = \alpha^2 > 0$, then $r = 1/4 \pm i\alpha$, so

$\phi(x) = e^{x/4}(c_1\cos(\alpha x) + c_2\sin(\alpha x))$, $\phi(0) = c_1 = 0$ $\phi(10) = e^{10/4}\cdot c_2\sin(10\alpha) = 0$

e.v.'s $\alpha_n = \frac{n\pi}{10}$ or $\lambda_n = \frac{1}{16} + \frac{n^2\pi^2}{100}$, e.f. $\phi_n(x) = e^{x/4}\sin\left(\frac{n\pi x}{10}\right)$, $n = 1, 2, \ldots$

c. Superposition $u(x,t) = \sum_{n=1}^{\infty} B_n e^{x/4}\sin\left(\frac{n\pi x}{10}\right)e^{-\left(\frac{1}{16} + \frac{n^2\pi^2}{100}\right)t}$

I.C. $u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n e^{x/4}\sin\left(\frac{n\pi x}{10}\right)$ by orthogonality

$$B_n = \frac{\int_0^{10} f(x)\, e^{x/4}\sin\left(\frac{n\pi x}{10}\right)e^{-0.5x}dx}{\int_0^{10}\left(e^{x/4}\sin\left(\frac{n\pi x}{10}\right)\right)^2 e^{-0.5x}dx} = \frac{1}{5}\int_0^{10} f(x)e^{-x/4}\sin\left(\frac{n\pi x}{10}\right)dx$$

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(3) $\frac{\partial^4 u}{\partial x^4} = -\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - k\frac{\partial u}{\partial t}$, B.C. $u(0,t)=0$, $\frac{\partial^2 u}{\partial x^2}(0,t)=0$, $u(a,t)=0$, $\frac{\partial^2 u}{\partial x^2}(a,t)=0$

I.C. $u(x,0)=0$, $\frac{\partial u}{\partial t}(x,0)=1$. Homogeneous B.C., Sep. of Var. $u(x,t)=\phi(x)h(t)$

3 $\phi''''h = -\frac{\phi}{c^2}h'' - k\phi h' \Rightarrow \frac{\phi^{(iv)}}{\phi} = -\frac{1}{c^2 h}(h''+kc^2 h) = \lambda$.

SL Prob $\phi^{(iv)} - \lambda\phi = 0$ with $\phi(0)=0$, $\phi''(0)=0$, $\phi(a)=0$ & $\phi''(a)=0$

If $\lambda=0$, $\phi(x)=Ax^3+Bx^2+Cx+D$. $\phi(0)=D=0$, $\phi''(0)=2B=0$, $\phi''(a)=6Aa=0$,

$(A=0)$, $\phi(a)=Ca=0$ $(C=0)$ only trivial soln.

If $\lambda=-4\alpha^4<0$, char. eqn $r^4=-4\alpha^4$, $r=\alpha(\pm1\pm i)$. It follows

$\phi(x) = A\cosh(\alpha x)\cos(\alpha x) + B\cosh(\alpha x)\sin(\alpha x) + C\sinh(\alpha x)\cos(\alpha x) + D\sinh(\alpha x)\sin(\alpha x)$

$\phi''(x) = 2\alpha^2(-A\sinh(\alpha x)\sin(\alpha x) + B\sinh(\alpha x)\cos(\alpha x) - C\cosh(\alpha x)\sin(\alpha x) + D\cosh(\alpha x)\cos(\alpha x))$

$\phi(0)=A=0$, $\phi''(0)=2\alpha^2 D=0 \Rightarrow D=0$, $\phi(a)=B\cosh(\alpha a)\sin(\alpha a)+C\sinh(\alpha a)\cos(\alpha a)=0$

$\phi''(a)=(B\sinh(\alpha a)\cos(\alpha a) - C\cosh(\alpha a)\sin(\alpha a))2\alpha^2=0$. From lin. alg. $B=C=0$ if

$\det\begin{vmatrix}\cosh(\alpha a)\sin(\alpha a) & \sinh(\alpha a)\cos(\alpha a)\\ \sinh(\alpha a)\cos(\alpha a) & -\cosh(\alpha a)\sin(\alpha a)\end{vmatrix} = -\cosh^2(\alpha a)\sin^2(\alpha a) - \sinh^2(\alpha a)\cos^2(\alpha a) \neq 0$

Since $\sin^2(\alpha a)=1-\cos^2(\alpha a)$, $\det|\cdot| = -\cosh^2(\alpha a) + \cos^2(\alpha a)(\cosh^2(\alpha a)-\sinh^2(\alpha a))$

$= -\cosh^2(\alpha a)+\cos^2(\alpha a) < 0$ for $\alpha a\neq 0$. ∴ only trivial soln.

If $\lambda=\alpha^4>0$, char. eqn. $r^4=\alpha^4$, $r=\pm\alpha, \pm i\alpha$. It follows

$\phi(x) = A\cosh(\alpha x)+B\sinh(\alpha x)+C\cos(\alpha x)+D\sin(\alpha x)$ and $\left.\begin{matrix}\phi(0)=A+C=0\\ \phi''(0)=\alpha^2(A-C)=0\end{matrix}\right\}\Rightarrow A=C=0$

$\phi''(x)=\alpha^2(A\cosh(\alpha x)+B\sinh(\alpha x)-C\cos(\alpha x)-D\sin(\alpha x))$,

$\phi(a)=B\sinh(\alpha a)+D\sin(\alpha a)=0$, $\phi''(a)=\alpha^2(B\sinh(\alpha a)-D\sin(\alpha a))=0 \Rightarrow B\sinh(\alpha a)=0$

or $B=0$ and $D\sin(\alpha a)=0$, which for $D\neq0$ gives $\sin(\alpha a)=0$, $\alpha a=n\pi$, $n=1,2\ldots$

12 E.V.'s $\lambda_n=\left(\frac{n\pi}{a}\right)^4$ E.F. $\phi_n(x)=\sin\left(\frac{n\pi x}{a}\right)$, $n=1,2,\ldots$

Time Eqn. $h''+kc^2h'+c^2\lambda_n h=0$. Char. Eqn. $r^2+kc^2r+c^2\lambda_n=0$, $r=-\frac{kc^2}{2}\pm\frac{1}{2}\sqrt{k^2c^4-4c^2\lambda_n}$

or $r=-\frac{kc^2}{2}\pm\frac{1}{2}\sqrt{k^2c^4-\frac{4c^2n^4\pi^4}{a^4}}$. With $k<\frac{2\pi^2}{a^2c}$, we define $\mu_n=\sqrt{c^2\lambda_n-\frac{k^2c^4}{4}}$

$h_n(t)=e^{-\frac{kc^2t}{2}}(A_n\cos(\mu_n t)+B_n\sin(\mu_n t))$, Superposition Principle gives

3 $u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{kc^2t}{2}}[A_n\cos(\mu_n t)+B_n\sin(\mu_n t)]$

2 From $u(x,0)=0$, it follows $A_n=0$.

$\frac{\partial u}{\partial t}(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{kc^2t}{2}}B_n\left[-\frac{kc^2}{2}\sin(\mu_n t)+\mu_n\cos(\mu_n t)\right]$

$\frac{\partial u}{\partial t}(x,0)=1=\sum_{n=1}^{\infty}B_n\mu_n\sin\left(\frac{n\pi x}{a}\right) \Rightarrow B_n=\frac{2}{a\mu_n}\int_0^a 1\cdot\sin\left(\frac{n\pi x}{a}\right)dx = -\frac{2}{n\pi\mu_n}\cos\left(\frac{n\pi x}{a}\right)\Big|_0^a$

5 ∴ $B_n=\frac{2}{n\pi\mu_n}(1-(-1)^n) \Rightarrow u(x,t)=\sum_{n=1}^{\infty}\frac{2(1-(-1)^n)}{n\pi\mu_n}\sin\left(\frac{n\pi x}{a}\right)e^{-\frac{kc^2t}{2}}\sin(\mu_n t)$

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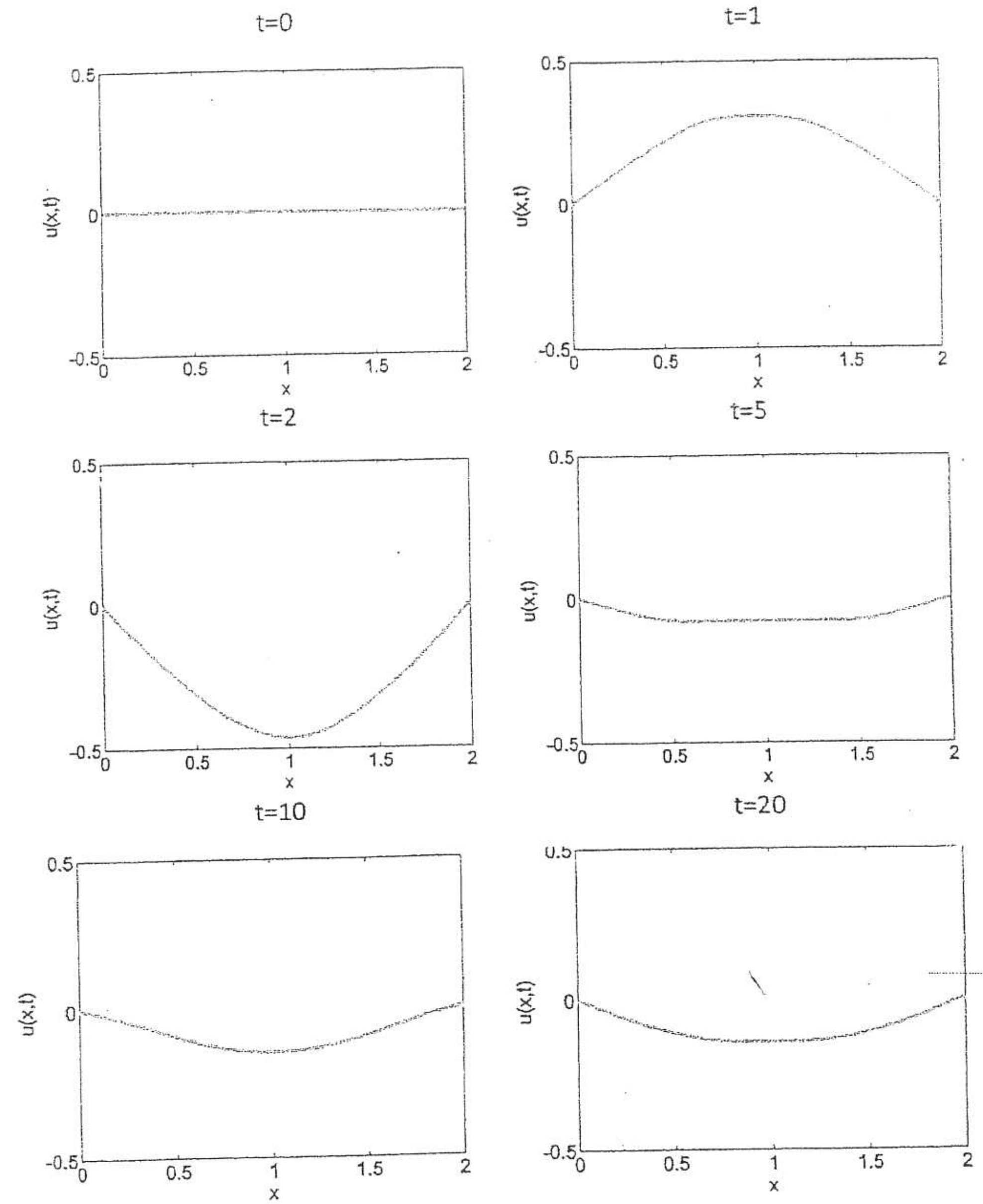

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0

4 Solve $\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left[\rho^2 \frac{\partial u}{\partial\rho}\right] = \frac{1}{k}\frac{\partial u}{\partial t}$, $u(0,t)$ bounded, $u(a,t) = 0$, $u(\rho,0) = T_0$, $0 < \rho < a$, $t > 0$. We let $u(\rho,t) = v(\rho,t)/\rho$, then $\partial u/\partial\rho = [\rho v_\rho - v]/\rho^2$. Then $\partial[\rho v_\rho - v]/\partial\rho = \rho v_{\rho\rho}$. Thus, $(1/\rho)v_{\rho\rho} = v_t/k\rho$, or $kv_{\rho\rho} = v_t$. We let $v(\rho,t) = R(\rho)T(t)$, then $kR''T = RT'$ or $R''/R = T'/kT = -\lambda$. The boundary conditions give $u(a,t) = v(a,t)/a = R(a)T(t)/a = 0$ which for nontrivial solutions gives $v(a) = 0$. $u(0,t)$ bounded implies $\lim_{\rho\to 0}\frac{v(\rho,t)}{\rho} = \lim_{\rho\to 0}\frac{R(\rho)T(t)}{\rho}$ is bounded or $R(\rho)/\rho$ is bounded as $\rho \to 0$. The Sturm-Liouville problem is given by $R'' + \lambda R = 0$ with $R(\rho)/\rho$ bounded as $\rho \to 0$ and $R(a) = 0$. Consider $\lambda = -\alpha^2 < 0$, then $R(\rho) = c_1\cosh(\alpha\rho) + c_2\sinh(\alpha\pi)$. $R(\rho)/\rho$ bounded as $\rho \to 0$ implies $c_1 = 0$. $R(a) = c_2\sinh(\alpha a) = 0$ implies $c_2 = 0$ which gives only the trivial solution. Now consider $\lambda = 0$, then $R(\rho) = c_1 + c_2\rho$. Boundedness implies $c_1 = 0$ and $R(a) = c_2 a = 0$ or $c_2 = 0$, so again we have the trivial solution. Finally let $\lambda = \gamma^2 > 0$, then $R(\rho) = c_1\cos(\gamma\rho) + c_2\sin(\gamma\pi)$. $R(\rho)/\rho$ bounded as $\rho \to 0$ implies $c_1 = 0$. $R(a) = c_2\sin(\gamma a) = 0$, which for nontrivial solutions has $\gamma_n = n\pi/a$, $n = 1,2,\ldots$ Thus the eigenvalues $\lambda_n = n^2\pi^2/a^2$ and the eigenfunctions $R_n(\rho) = \sin(n\pi\rho/a)$. Next we solve the T equation, $T' + k(n^2\pi^2/a^2)T = 0$, or $T_n(t) = \exp[-kn^2\pi^2 t/a^2]$. The product solution is given by $v_n(\rho,t) = \sin(n\pi\rho/a)\exp[-kn^2\pi^2 t/a^2]$. We take a linear combination and find that

$$v(\rho,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi\rho/a)\exp[-kn^2\pi^2 t/a^2].$$

Finally, we must satisfy the initial condition which as $u(\rho,0) = v(\rho,0)/\rho = T_0$, gives $T_0\rho = \sum_{n=1}^{\infty} b_n\sin(n\pi\rho/a)$.

Hence, $b_n = \frac{2}{a}\int_0^a T_0\rho\sin(n\pi\rho/a)d\rho = -(2T_0 a/n\pi)\cos(n\pi)$. With this information we obtain the solution to our original problem and it is given by

$$u(\rho,t) = -2T_0 a\sum_{n=1}^{\infty}\frac{\cos(n\pi)\sin(n\pi\rho/a)}{n\pi\rho}\exp[-kn^2\pi^2 t/a^2].$$

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(5) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$, $0 < x, y, z < 2$ with B.C.:

$\frac{\partial u}{\partial x}(0,y,z) = 0$, $u(2,y,z) = 0$, $u(x,0,z) = T_0$, $\frac{\partial u}{\partial y}(x,2,z) = 0$

$u(x,y,0) = 0$, $-K\frac{\partial u}{\partial z}(x,y,2) = h\,u(x,y,2)$

All but $y=0$ B.C.'s are homogeneous. Use sep. of variables $u(x,y,z) = X(x)Y(y)Z(z)$

$\therefore X''YZ + XY''Z + XYZ'' = 0 \Rightarrow \frac{X''}{X} = -\left(\frac{Y''}{Y} + \frac{Z''}{Z}\right) = p$ The B.C. at $x=0,2$

$\Rightarrow X'(0) = 0$ and $X(2) = 0$. If $p = \alpha^2 > 0$, $X(x) = A\cosh\alpha x + B\sinh\alpha x$.

$X'(0) = 0 = \alpha B$, $X(2) = 0 = A\cosh(\alpha 2)$ $\therefore$ only trivial soln. If $p = 0$, $X(x) = Ax + B$

$X'(0) = 0 = A$ $X(2) = 0 = B \Rightarrow$ only trivial soln. $\therefore$ Let $p = -\lambda^2 < 0$,

$X(x) = A\cos\lambda x + B\sin\lambda x$. $X'(0) = B\lambda = 0$. $X(2) = A\cos 2\lambda = 0$. For $A \neq 0$, $\lambda_n = \frac{(2n-1)\pi}{4}$

$n = 1, 2, \ldots$ e.v.'s with corresponding e.f.'s $X_n(x) = \cos\lambda_n x$.

Now return to $-\left(\frac{Y''}{Y} + \frac{Z''}{Z}\right) = -\lambda_n^2 \Rightarrow \frac{Z''}{Z} = -\frac{Y''}{Y} + \lambda_n^2 = q$. The B.C. at $z=0,2$

$\Rightarrow Z(0) = 0$ and $-KZ'(2) = hZ(2)$. If $q = \beta^2 > 0$, $Z(z) = A\cosh\beta z + B\sinh\beta z$

$Z(0) = 0 = A$. $-BK\beta\cosh(\beta 2) = hB\sinh(\beta 2) \Rightarrow B = 0$ $\therefore$ only trivial soln. If $q = 0$

$Z(z) = Az + B$. $Z(0) = 0 = B$ $-KA = 0 \Rightarrow A = 0$ $\therefore$ only trivial soln. Let $q = -\mu^2 < 0$

$Z(z) = A\cos\mu z + B\sin\mu z$. $Z(0) = A = 0$. $-K\mu B\cos 2\mu = hB\sin 2\mu$. For $B \neq 0$,

take μ_m to solve $\mu = -\frac{h}{K}\tan 2\mu$. Infinitely many μ_m, $m = 1, 2, \ldots$ with corresponding e.f.'s $Z_m(z) = \sin\mu_m z$.

Now Y_{mn} solves $Y'' - \lambda_{mn}^2 Y = 0$ where $\lambda_{mn}^2 = \lambda_n^2 + \mu_m^2$

$\Rightarrow Y_{mn}(y) = a_{mn}\cosh\lambda_{mn}y + b_{mn}\cosh\lambda_{mn}(2-y)$

By superposition principle

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos\lambda_n x\,\sin\mu_m z\left(a_{mn}\cosh\lambda_{mn}y + b_{mn}\cosh\lambda_{mn}(2-y)\right)$$

$$\frac{\partial u}{\partial y}(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\cos\lambda_n x\,\sin\mu_m z\left(a_{mn}\lambda_{mn}\sinh\lambda_{mn}y - b_{mn}\lambda_{mn}\sinh\lambda_{mn}(2-y)\right)$$

From B.C. at $y = 2$

$$\frac{\partial u}{\partial y}(x,2,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\lambda_{mn}\sinh(\lambda_{mn}2)\cos\lambda_n x\,\sin\mu_m z = 0 \Rightarrow a_{mn} = 0$$

From B.C. at $y = 0$

$$u(x,0,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}\cosh(\lambda_{mn}2)\cos\lambda_n x\,\sin\mu_m z = T_0$$

First mult. by $\sin\mu_i z$ + integrate 0 to 2 (using orthogonality)

$$\sum_{n=1}^{\infty} b_{in}\cosh 2\lambda_{in}\cos\lambda_n x = T_0 \int_0^2 \sin\mu_i z\,dz \Big/ \int_0^2 \sin^2\mu_i z\,dz = \frac{2h(1-\cos 2\mu_i)}{\mu_i(2h+K\cos^2 2\mu_i)}T_0 \text{ , as}$$

$$\int_0^2 \sin\mu_i z\,dz = -\frac{\cos\mu_i z}{\mu_i}\Big|_0^2 = \frac{1}{\mu_i}\left[1-\cos 2\mu_i\right] \text{ , and } \int_0^2 \sin^2\mu_i z\,dz = \left[\frac{z}{2} - \frac{\sin 2\mu_i z}{4\mu_i}\right|_0^2 = 1 - \frac{\sin 4\mu_i}{4\mu_i}$$

$= 1 + \frac{K}{2h}\cos^2 2\mu_i$. Now mult. by $\cos\lambda_j x$ + integrate 0 to 2 w.r.t. x.

$$b_{ij}\cosh 2\lambda_{ij} = T_0 \frac{2h(1-\cos 2\mu_i)}{\mu_i(2h+K\cos^2 2\mu_i)} \frac{\int_0^2 \cos\lambda_j x\,dx}{\int_0^2 \cos^2\lambda_j x\,dx} \text{ , but}$$

$$\int_0^2 \cos\lambda_j x\,dx = \frac{\sin\lambda_j x}{\lambda_j}\Big|_0^2 = \frac{1}{\lambda_j}\sin\frac{(2j-1)}{2}\pi \text{ , and } \int_0^2 \cos^2\lambda_j x\,dx = 1$$

$$\therefore b_{ij} = T_0 \frac{2h(1-\cos 2\mu_i)\sin\left(\frac{2j-1}{2}\right)\pi}{\mu_i\lambda_j(2h+K\cos^2 2\mu_i)\cosh 2\lambda_{ij}} \qquad = (-1)^{j+1}$$

and $$u(x,y,z) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} b_{ij}\cos\lambda_j x\,\sin\mu_i z\,\cosh\lambda_{ij}(2-y)$$

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6 (a) $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0$, $0<r<1$, $0<z<4$

$u(1,z)=0$, $u(r,0)=0$, $u(r,4)=20$

$u(0,z)$ BOUNDED

SEPARATION OF VARIABLES

$u(r,z) = \phi(r)\,H(z)$

$\frac{1}{r}\frac{\partial}{\partial r}(r\,\phi' H) + \phi H'' = 0$ DIVIDING BY ϕH

$\frac{1}{\phi}\frac{1}{r}\frac{d}{dr}(r\phi') + \frac{1}{H}H'' = 0$

$\frac{1}{r\phi}\frac{d}{dr}(r\phi') = -\frac{1}{H}H'' = -\lambda$

GIVES TWO DIFF EQUATIONS:

① $H'' = \lambda H$

② $\frac{d}{dr}(r\phi') = -\lambda r\phi$

② $\frac{d}{dr}(r\phi') + \lambda r\phi = 0$ SL EQN W/ $p(r)=r$, $q(r)=0$, $\sigma(r)=r$

W/ $\phi(0)$ BOUNDED

$\phi(1)=0$

RAYLEIGH QUOTIENT SHOWS

$\lambda = \min \frac{-r\phi'\phi\big|_0^1 + \int_0^1 r(\phi')^2\,dr}{\int_0^1 r\phi^2\,dr} = \frac{-[1\,\phi'(1)\phi(1) - 0] + (\text{pos})}{\text{pos}}$

$=$ pos , $\phi' \neq 0$ OTHERWISE $\phi = c$ B/C $\phi(1)=c=0 \Rightarrow c=0$

THUS, $\lambda > 0$ AND WE CAN SUBSTITUTE $z=\sqrt{\lambda}\,r$

② BECOMES $\frac{d}{dz}\left(z\frac{d\phi}{dz}\right) + z\phi = 0$

$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + z^2\phi = 0$

BESSEL'S DIFF EQN OF ORDER $m=0$

Aside

SOL'N IS $\phi(r) = c_1 J_0(\sqrt{\lambda}\,r) + c_2 Y_0(\sqrt{\lambda}\,r)$

BUT SINCE $\phi(0)$ BOUNDED, $c_2 = 0$

$\Rightarrow \phi(r) = J_0(\sqrt{\lambda}\,r)$ WHERE λ'S ARE DETERMINED BY

$J_0(\sqrt{\lambda}) = 0 \Rightarrow \lambda_{0n} = (z_{0n})^2$ ZEROS OF J_0

① $H'' = \lambda H \Rightarrow H(z) = c_1\cosh(\sqrt{\lambda}\,z) + c_2\sinh(\sqrt{\lambda}\,z)$

THE HOMOGENEOUS BC GIVES:

$H(0) = 0 = c_1$

THUS $H(z) = \sinh(\sqrt{\lambda}\,z)$

CONT → (9)

$$u(r,z,t) = v(r,z,t) + u_E(r,z)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\, J_0(\sqrt{\gamma_m}\, r)\, \sin\!\left(\frac{n\pi z}{4}\right) e^{-\lambda_{mn} t}$$

$$+ \sum_{m=1}^{\infty} A_m J_0(\sqrt{\gamma_m}\, r)\, \sinh(\sqrt{\gamma_m}\, z)$$

where $\gamma_m = (z_{0m})^2$ zeros of [illegible]

$$A_m = 20 \int_0^1 r J_0(\sqrt{\gamma_m}\, r)\, dr \Big/ \left[\sinh(4\sqrt{\gamma_m}) \int_0^1 r J_0^2(\sqrt{\gamma_m}\, r)\, dr\right]$$

$$B_{mn} = \frac{1}{2}\int_0^4\int_0^1 \underline{(30 - u_E(r,z))}\, r J_0(\sqrt{\gamma_m}\, r)\, \sin\!\left(\frac{n\pi z}{4}\right) dr\, dz \Big/ \int_0^1 r J_0^2(\sqrt{\gamma_m}\, r)\, dr$$

$$\lambda_{mn} = z_{0m}^2 - \left(\frac{n\pi}{4}\right)^2$$

$$u_E(r,z) = \sum_{m=1}^{\infty} A_m J_0(\sqrt{\gamma_m}\, r)\, \sinh(\sqrt{\gamma_m}\, z)$$

reduces by orthogonality

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#2 Extra Credit

$$X := evalf\left(\int_0^{2\cdot\pi}\int_0^1\int_0^4 \sum_{i=1}^{50} \frac{\int_0^1 20\cdot r\cdot \mathrm{BesselJ}(0, r\cdot \mathrm{BesselJZeros}(0,i))\, dr}{\sinh(4\cdot \mathrm{BesselJZeros}(0,i))\cdot \int_0^1 r\cdot(\mathrm{BesselJ}(0, r\cdot \mathrm{BesselJZeros}(0,i)))^2\, dr} \cdot \sinh(z\cdot \mathrm{BesselJZeros}(0,i))\cdot \mathrm{BesselJ}(0, r\cdot \mathrm{BesselJZeros}(0,i))\cdot r\, dz\, dr\, d\theta\right)$$

20.32372562 **(1)**

#This is the total heat. Notice the inclusion of the extrea r term since we are integrating **in** *cylindrical coordinates. Now divide* **by** *the volume* **to** *get the average temperature.*

Bonus 5

$$evalf\left(\frac{X}{4\cdot\pi}\right)$$

1.617310697 **(2)**

B) $\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right)$ BC: $u(1,z,t)=0,\ u(r,0,t)=0,\ u(r,4,t)=20$

IC: $u(r,z,0)=20$

LET $u(r,z,t) = v(r,z,t) + u_E(r,z)$ FROM PART (A)

THEN $v = u - u_E$ WITH $v(1,z,t)=0$, $v(r,0,t)=0$, $v(r,4,t)=0$

NOW HOMOGENEOUS B

AND $v(r,z,0) = 20 - u_E$

AND $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial u_E}{\partial t}^{0}$

$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} + \frac{\partial u_E}{\partial r}$

$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 u_E}{\partial z^2}$

WHERE u_E SATISFIES $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_E}{\partial r}\right) + \frac{\partial^2 u_E}{\partial z^2} = 0$

THE ORIGINAL PDE BECOMES:

$$\frac{\partial v}{\partial t} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r} + r\frac{\partial u_E}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 u_E}{\partial z^2}\right]$$

$$= k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_E}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 u_E}{\partial z^2}\right]$$

IS EQUAL TO ZERO

THUS, $\frac{\partial v}{\partial t} = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2}\right]$

SEPARATION of VARIABLES

$v(r,z,t) = \phi(r,z)\,G(t)$

$\Rightarrow \phi G' = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}G\right) + \frac{\partial^2\phi}{\partial z^2}G\right]$ DIVIDING BY $\phi G k$

$\frac{G'}{kG} = \frac{1}{\phi}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2}\right] = -\lambda$

WHERE $\lambda > 0$ B/C THE TIME DEPENDENT PART NEEDS TO DECAY (NOT GROW)

WE get TWO DIFF EQNS.

① $G' = -\lambda k G \Rightarrow \left[\, G(t) = c_1 e^{-\lambda k t} \,\right]$

② $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2} = -\lambda\phi$

② SEPARATION of VARIABLES $\phi(r,z) = F(r)H(z)$

$\frac{1}{r}\frac{d}{dr}(rHF') + FH'' = -\lambda FH$ DIVIDING BY FH

$\frac{1}{Fr}\frac{d}{dr}(rF') + \frac{1}{H}H'' = -\lambda$

$\frac{1}{Fr}\frac{d}{dr}(rF') + \lambda = -\frac{1}{H}H'' = \mu$

GIVES TWO DIFF EQNS

③ $H'' = -\mu H$ w/ $H(0)=0$, $H(4)=0$

④ $\frac{1}{Fr}\frac{d}{dr}(rF') + (\lambda-\mu) = 0$ w/ $F(0)$ BOUNDED, $F(1)=0$

CONT → ⑥

③ $H'' = -\mu H$, $H(0) = 0$, $H(4) = 0$

WHERE $\mu > 0$, PROOF :

* IF $\mu = 0$, $H'' = 0 \Rightarrow H(z) = c_1 z + c_2$

BC'S IMPLY $H(0) = 0 = c_2$

$H(4) = 0 = 4c_1 \Rightarrow c_1 = 0$

THUS, WE GET THE TRIVIAL SOLUTION $H(z) = 0$

* IF $\mu = -\alpha^2 < 0$

$H(z) = c_1 \cosh(z\alpha) + c_2 \sinh(z\alpha)$

THE BC'S IMPLY

$H(0) = 0 = c_1$

$H(4) = 0 = c_2 \sinh(4\alpha)$

$\Rightarrow 4\alpha = in\pi$, $n = 1, 2, 3 \ldots$

$\alpha = \frac{in\pi}{4}$

$\mu = -\alpha^2 = -\left(\frac{in\pi}{4}\right)^2 = \left(\frac{n\pi}{4}\right)^2 > 0$ CONTRADICTION

THUS, μ CANNOT BE < 0

* FOR $\mu > 0$

$H(z) = c_1 \cos(z\sqrt{\mu}) + c_2 \sin(z\sqrt{\mu})$

THE BOUNDARY CONDITIONS GIVE

$H(0) = 0 = c_1$

$H(4) = 0 = c_2 \sin(4\sqrt{\mu})$

$\Rightarrow 4\sqrt{\mu} = n\pi$, $n = 1, 2, 3 \ldots$

$\sqrt{\mu} = \frac{n\pi}{4}$

$\left[H(z) = c_2 \sin\left(\frac{n\pi z}{4}\right) , \quad n = 1, 2, 3 \ldots \right]$

④ $\frac{d}{dr}\left(r \frac{dF}{dr}\right) + \overbrace{(\lambda - \mu_n)}^{\gamma} Fr = 0$, $F(0)$ BOUNDED, $F(1) = 0$

IS A SL EQUATION W/ $p(r) = r$, $q(r) = 0$, $\sigma(r) = r$

$$\gamma = \text{MIN} \frac{-rF'F\big|_0^1 + \int_0^1 r(F')^2\,dr}{\int_0^1 rF^2\,dr} = \frac{[-F'(1)\overset{\to 0}{F(1)} - 0] + \int_0^1 r(F')^2\,dr}{\int_0^1 rF^2\,dr}$$

THUS, IF $\int_0^1 r(F')^2\,dr = 0 \Rightarrow F'(r) = 0$

$\Rightarrow F(r) = \text{CONST}$

BUT $F(1) = 0 = \text{CONST} \Rightarrow F(r) = 0$

THUS, $\gamma > 0$

$$\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \gamma F r = 0 \qquad (*)$$

SINCE $\gamma > 0$, WE CAN LET $z = \sqrt{\gamma}\, r$

THEN $(*)$ BECOMES

$$\frac{d}{dz}\left(z\frac{dF}{dz}\right) + zF = 0 \Rightarrow z^2\frac{d^2F}{dz^2} + z\frac{dF}{dz} + z^2 F = 0$$

WHICH IS BESSEL'S DIFF EQN of ORDER ZERO.

THUS, $F(r) = c_1 J_0(\sqrt{\gamma}\, r) + c_2 Y_0(\sqrt{\gamma}\, r)$

SINCE $F(0)$ BOUNDED, $c_2 = 0$

$\Rightarrow F(r) = c_1 J_0(\sqrt{\gamma}\, r)$

THE BOUNDARY CONDITION IMPLIES

$F(1) = 0 = J_0(\sqrt{\gamma}) \Rightarrow \sqrt{\gamma} = z_{om}$

ZERO's of BESSEL'S FUNCTION

$$\gamma_m = \lambda - \mu_n = z_{om}^2$$

$$\Rightarrow \lambda_{mn} = z_{om}^2 + \mu_n$$

$$\left[F(r) = J_0(r\sqrt{\gamma_{mn}})\right]$$

BY THE SUPERPOSITION PRINCIPLE

$$V(r,z,t) = F(r)H(z)G(t)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_0(r\sqrt{\gamma_{mn}}) \sin\left(\frac{n\pi z}{4}\right) e^{-\lambda_{mn} kt}$$

WHERE $\lambda_{mn} = z_{om}^2 + \left(\frac{n\pi}{4}\right)^2$, $\gamma_{mn} = z_{om}^2$

ZERO's of BESSEL'S FCN

THE INITIAL CONDITION IS $V(r,z,0) = 20 - u_E$

$$\Rightarrow 20 - u_E = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_0(r\sqrt{\gamma_{mn}}) \sin\left(\frac{n\pi z}{4}\right)$$

USING ORTHOGONALITY of J_0 & SIN

$$\int_0^4\int_0^1 (20 - u_E(r,z))\, r J_0(r\sqrt{\gamma_{mn}}) \sin\left(\frac{n\pi z}{4}\right) dr\, dz$$

$$= B_{mn}\int_0^1\int_0^4 r J_0^2(r\sqrt{\gamma_{mn}}) \sin^2\left(\frac{n\pi z}{4}\right) dz\, dr$$

$$\left(\frac{4}{2}\right) = 2$$

$$B_{mn} = \frac{1}{2}\int_0^4\int_0^1 (20 - u_E(r,z))\, r J_0(r\sqrt{\gamma_{mn}}) \sin\left(\frac{n\pi z}{4}\right) dr\, dz \Big/ \int_0^1 r J_0^2(r\sqrt{\gamma_{mn}})\, dr$$

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