Sp 12 Math 342 B Take-Home Final

① a. $f(x) = e^{x/2}$ Find $f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x/2}\,dx = \frac{1}{\pi}(e^{\pi/2} - e^{-\pi/2}), \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x/2}\cos(nx)\,dx = \frac{2(-1)^n}{\pi(1+4n^2)}[e^{\pi/2} - e^{-\pi/2}]$$

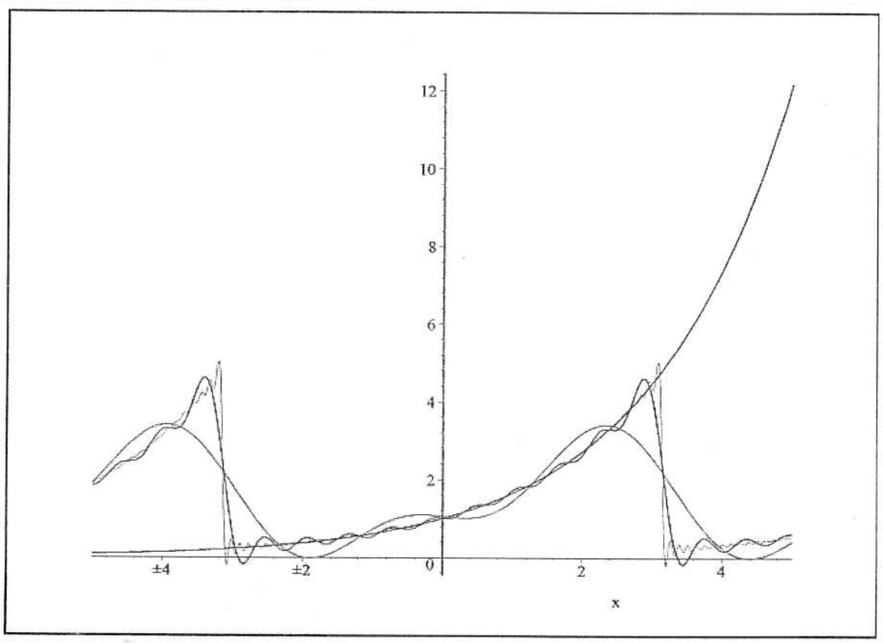
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x/2}\sin(nx)\,dx = \frac{4n(-1)^n}{\pi(1+4n^2)}[e^{-\pi/2} - e^{\pi/2}]$$

Thus,

$$f(x) \sim \frac{e^{\pi/2} - e^{-\pi/2}}{\pi}\left[1 + \sum_{n=1}^{\infty}\left(\frac{2(-1)^n}{1+4n^2}\cos(nx) - \frac{4n(-1)^n}{1+4n^2}\sin(nx)\right)\right]$$

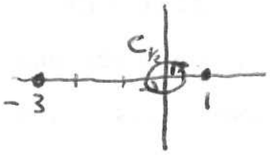
b. The Fourier series is 2π-periodic. By the convergence Th we have the Fourier series converges to $f(0) = 1$ at $x = 0$. It converges to the midpt. at $x = \pi$, so $f(x) \sim \frac{f(\pi) + f(-\pi)}{2} = \frac{1}{2}(e^{\pi/2} + e^{-\pi/2})$. By periodicity, the series at $x = 5$ converges to $f(5 - 2\pi) = e^{\frac{5}{2} - \pi} \approx 0.52645$

c. Below is a plot of $f(x)$ and the Fourier series with $n = 2, 10,$ and 50 for $x \in [-5, 5]$.



(2) Compute the contour integral $\oint_C \frac{e^z}{z^2+2z-3}\,dz = \oint_C \frac{e^z}{(z+3)(z-1)}\,dz = \oint_C f(z)\,dz$

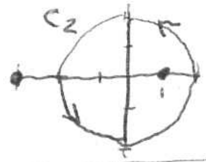
a. $C_{1/2}$ is circle radius $1/2$ centered at origin.



Since $f(z)$ is analytic inside and on C

$$\oint_{C_{1/2}} f(z)\,dz = 0.$$

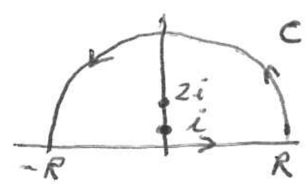
b. C_2 is circle radius 2 centered at origin.



From Residue Th with a simple pole at $z=1$

$$\oint_{C_2} f(z)\,dz = 2\pi i\, R(1) = 2\pi i \frac{e^1}{4} = \frac{e\pi i}{2}.$$

(3) Evaluate $\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}\,dx = \int_0^\infty \frac{2x^2-1}{(x^2+4)(x^2+1)}\,dx$



Consider the contour C with $R>2$.

The fcn is even, so $\int_{-\infty}^{\infty} f(x)\,dx = 2\int_0^\infty f(x)\,dx$

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \oint_{C_R} f(z)\,dz = 2\pi i\big(R(i) + R(2i)\big)$$

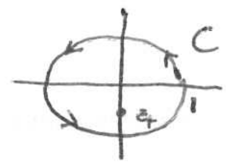
$$R(i) = \lim_{z\to i} \frac{(z-i)(2z^2-1)}{(z^2+4)(z+i)(z-i)} = \frac{-3}{3(2i)} = \frac{i}{2}$$

$$R(2i) = \lim_{z\to 2i} \frac{(z-2i)(2z^2-1)}{(z+2i)(z-2i)(z^2+1)} = \frac{-9}{4i(-3)} = -\frac{3i}{4}$$

$$\oint_{C_R} \frac{2z^2-1}{(z^2+4)(z^2+1)}\,dz \le \pi R \frac{2R^2+1}{(R^2-4)(R^2-1)},$$ which tends to 0 as $R\to\infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left(\frac{i}{2} - \frac{3i}{4}\right) = \frac{\pi}{2},$$ so $$\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}\,dx = \frac{\pi}{4}$$

(4) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin(\theta)}$ with $a \in (0,1)$.



$z = e^{i\theta}$, $dz = ie^{i\theta}\,d\theta$, $\sin(\theta) = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin(\theta)} = \oint_C \frac{1}{1+a\left(\frac{z-1/z}{2i}\right)}\frac{dz}{iz} = \oint_C \frac{2/a\,dz}{z^2+\frac{2i}{a}z-1}$$

Denominator satisfies $z_\pm = \dfrac{-i \pm \sqrt{a^2-1}}{a} = i\left(\dfrac{-1\pm\sqrt{1-a^2}}{a}\right)$ (purely imaginary roots)

Clearly $|z_-| = \left|i\left(\frac{-1-\sqrt{1-a^2}}{a}\right)\right| > 1$, lies outside C, and $|z_+ \cdot z_-| = 1$, so

z_+ must lie inside C (a simple pole). By the Residue Th

$$\oint_C \frac{2/a\,dz}{z^2+\frac{2i}{a}z-1} = 2\pi i\, R\left(i\frac{(-1+\sqrt{1-a^2})}{a}\right) = 2\pi i \frac{2/a}{z_+-z_-}$$

$$= 2\pi i \frac{2/a}{\frac{i}{a}\left(\sqrt{1-a^2}+\sqrt{1-a^2}\right)} = \frac{2\pi}{\sqrt{1-a^2}} = \int_0^{2\pi}\frac{d\theta}{1+a\sin(\theta)}$$

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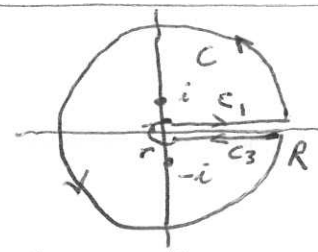
(5) Evaluate $\displaystyle\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx$. Consider



a. $$\oint_C \frac{z^{1/2}}{1+z^2}dz = 2\pi i\left(R(i)+R(-i)\right)$$

$$R(i) = \lim_{z\to i}\frac{(z-i)z^{1/2}}{1+z^2} = \frac{i^{1/2}}{2i} = \frac{1}{2}i^{-1/2}, \quad R(-i) = \lim_{z\to -i}\frac{(z+i)z^{1/2}}{1+z^2} = \frac{(-i)^{1/2}}{-2i} = -\frac{1}{2}i^{1/2}$$

$$\oint_C \frac{z^{1/2}}{1+z^2}dz = 2\pi i\left(\frac{1}{2}\left(\frac{1-i}{\sqrt{2}}\right) - \left(\frac{1+i}{\sqrt{2}}\right)\right) = \pi\sqrt{2}$$

Let $z = Re^{i\theta}$ $dz = iRe^{i\theta}d\theta$

$$\oint_{C_R}\frac{z^{1/2}}{1+z^2}dz = \int_0^{2\pi}\frac{R^{1/2}e^{i\theta/2}}{1+R^2e^{i2\theta}}iRe^{i\theta}d\theta,$$ but for $R\to\infty$, the integrand

is approximately $1/R^{1/2} \to 0$. Thus, integral $\to 0$ along C_R.

Similarly if $z = re^{i\theta}$ $dz = ire^{i\theta}d\theta$

$$\oint_{C_r}\frac{z^{1/2}}{1+z^2}dz = \int_0^{2\pi}\frac{r^{1/2}e^{i\theta/2}}{1+r^2e^{i2\theta}}ire^{i\theta}d\theta,$$ but for $r\to 0$, the integrand

is approximately $r^{3/2}\to 0$. Thus, integral $\to 0$ along C_r.

$$\int_\infty^0 \frac{(re^{2\pi i})^{1/2}e^{2\pi i}}{1+r^2e^{4\pi i}}dr = -\int_0^\infty\frac{r^{1/2}e^{\pi i}}{1+r^2}dr = \int_0^\infty\frac{r^{1/2}}{1+r^2}dr$$

$$\oint_C\frac{z^{1/2}}{1+z^2}dz = \pi\sqrt{2} = \oint_{C_1}f(z)dz + \oint_{C_R}f(z)dz + \oint_{C_3}f(z)dz + \oint_{C_r}f(z)dz \to 2\int_0^\infty\frac{x^{1/2}}{1+x^2}dx$$

5 a (cont.) Thus,

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \frac{\pi\sqrt{2}}{2}$$

b. Let $u = x^2$, $du = 2x\,dx$, $x = u^{1/2}$

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \int_0^\infty \frac{u^{1/4}}{1+u}\cdot\frac{du}{2u^{1/2}} = \frac{1}{2}\int_0^\infty \frac{u^{-1/4}}{1+u}\,du = \frac{1}{2}B\left(\tfrac{3}{4},\tfrac{1}{4}\right)$$

$p-1 = -1/4$, $p = 3/4$, $q = 1/4$ $\quad \left(B(p,q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}\,dy \right)$

c. $$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \frac{1}{2}B\left(\tfrac{3}{4},\tfrac{1}{4}\right) = \frac{1}{2}\frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(1)} = \frac{1}{2}\Gamma(3/4)\Gamma(1/4)$$

but $\Gamma(1/4) = \frac{\pi\sqrt{2}}{\Gamma(3/4)} \Rightarrow \int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \frac{\pi\sqrt{2}}{2}$

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6 Consider $y'' - xy' - y = 0$. Let $y(x) = \sum_{n=0}^\infty a_n x^n$

$y' = \sum_{n=1}^\infty a_n n x^{n-1}$, $y'' = \sum_{n=2}^\infty a_n n(n-1)x^{n-2}$

$$\sum_{n=2}^\infty a_n n(n-1)x^{n-2} - \sum_{n=1}^\infty a_n n x^n - \sum_{n=0}^\infty a_n x^n = 0$$

$$\sum_{n=0}^\infty a_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^\infty a_n n x^n - \sum_{n=0}^\infty a_n x^n = 0$$

The constants a_0 and a_1 are arbitrary. $n=0 \Rightarrow a_2 \cdot 2 = a_0$ or $a_2 = a_0/2$. Recurrence relation:

$$a_{n+2}(n+2)(n+1) = a_n(n+1) \Rightarrow a_{n+2} = \frac{a_n}{n+2}$$

Two solns are

$$y_1(x) = a_0\left(1 + \frac{x^2}{2} + \frac{x^4}{2\cdot 4} + \frac{x^6}{2\cdot 4\cdot 6} + \ldots\right) = a_0 \sum_{n=0}^\infty \frac{x^{2n}}{2^n n!}$$

$$y_2(x) = a_1\left(x + \frac{x^3}{3} + \frac{x^5}{3\cdot 5} + \ldots\right) = a_1 \sum_{n=0}^\infty \frac{2^n n!\, x^{2n+1}}{(2n+1)!}$$

General soln.

$$y(x) = y_1(x) + y_2(x)$$

From ratio test on y_1 $\quad \lim_{n\to\infty} \frac{\left|\frac{x^{2(n+1)}}{2^{n+1}(n+1)!}\right|}{\left|\frac{x^{2n}}{2^n n!}\right|} = \lim_{n\to\infty} \frac{x^2}{2(n+1)} \to 0$ for all x

$y_1(x)$ converges for $|x| < \infty$.

$$\lim_{n\to\infty} \frac{\left|\frac{2^{n+1}(n+1)!\,x^{2n+3}}{(2n+3)!}\right|}{\left|\frac{2^n n!\,x^{2n+1}}{(2n+1)!}\right|} = \lim_{n\to\infty} \frac{2(n+1)x^2}{2(n+1)(2n+3)} = \lim_{n\to\infty}\frac{x^2}{2n+3} \to 0 \text{ for all } x$$

$y_2(x)$ converges for all x.

7 Consider $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $0 < x < 2$, $0 < y < 4$

B.C. $\frac{\partial u}{\partial x}(0,y) = 0$, $u(2,y) = 0$, $u(x,0) = 3 + 4\cos\left(\frac{7\pi}{4}\right)$, $u(x,4) = 0$.

Let $u(x,y) = X(x)\,Y(y)$, then $X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$.

SL Prob $X'' + \lambda X = 0$ with $X'(0) = 0$ and $X(2) = 0$. Clearly $\lambda > 0$

Let $\lambda = \alpha^2 > 0$, $X(x) = c_1\cos(\alpha x) + c_2\sin(\alpha x)$, $X'(x) = -c_1\alpha\sin(\alpha x) + c_2\alpha\cos(\alpha x)$

$X'(0) = 0 \Rightarrow c_2 = 0$. $X(2) = c_1\cos(2\alpha) = 0$. For nontrivial soln. take

$2\alpha = n\pi - \pi/2$ or $\alpha_n = \frac{(2n-1)\pi}{4}$, $n = 1, 2, \ldots$

Thus, e.v.'s $\lambda_n = \frac{(2n-1)^2\pi^2}{16}$ with e.f. $X_n(x) = \cos\left(\frac{(2n-1)\pi x}{4}\right)$

Y eqn $Y_n'' - \frac{(2n-1)^2\pi^2}{16} Y_n = 0$, $Y(4) = 0$. Take soln.

$Y_n(y) = c_1\cosh(\alpha_n(4-y)) + c_2\sinh(\alpha_n(4-y))$. $Y_n(4) = 0 \Rightarrow c_1 = 0$

Superposition Principle:

$$u(x,y) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{4}\right)\sinh\left(\frac{(2n-1)\pi}{4}(4-y)\right)$$

Non homogeneous B.C.

$$u(x,0) = 3 + 4\cos\left(\frac{7\pi}{4}\right) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{4}\right)\sinh((2n-1)\pi)$$

Except for $n=4$, $a_n = \frac{2}{2}\int_0^2 3\cos\left(\frac{(2n-1)\pi x}{4}\right)dx \Big/ \sinh((2n-1)\pi) = \frac{12(-1)^{n+1}}{(2n-1)\pi\sinh((2n-1)\pi)}$

When $n=4$ $a_4 = \left(4 - \frac{12}{7\pi}\right)\Big/\sinh(7\pi)$

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8 Consider $\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$ $0 < r < 1$, $0 < \theta < \frac{\pi}{2}$, $t > 0$

B.C. $u(1,\theta,t) = 0$, $u(r,0,t) = 0$, $u_\theta(r, \pi/2, t) = 0$ I.C. $u(r,\theta,0) = T_0 r^3\sin(3\theta)$

Let $u(r,\theta,t) = R(r)H(\theta)T(t)$, then $RHT' = kT\left(\frac{H}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}H''\right)$

$\Rightarrow \frac{T'}{kT} = \frac{\frac{1}{r}\frac{d}{dr}(rR')}{R} + \frac{1}{r^2}\frac{H''}{H} = -\lambda$ $T' + \lambda kT = 0 \Rightarrow T(t) = c\,e^{-\lambda kt}$

$\frac{r\frac{d}{dr}(rR')}{R} + \lambda r^2 = -\frac{H''}{H} = \mu$. First SL Prob: $H'' + \mu H = 0$, $H(0) = 0$, $H'(\frac{\pi}{2}) = 0$

Clearly $\mu > 0$, so let $\mu = \alpha^2 > 0$ $H(\theta) = c_1\cos(\alpha\theta) + c_2\sin(\alpha\theta)$. $H(0) = 0 \Rightarrow$

$c_1 = 0$. $H'(\theta) = c_2\alpha\cos(\alpha\theta)$. $H'(\frac{\pi}{2}) = c_2\alpha\cos(\alpha\frac{\pi}{2}) = 0 \Rightarrow \alpha = 1, 3, 5, \ldots$

e.v. $\mu_m = m^2$, e.f. $H_m(\theta) = \sin(m\theta)$, $m = 1, 3, 5, \ldots$

Second SL Prob $\frac{d}{dr}(rR') + \left(\lambda r - \frac{m^2}{r}\right)R = 0$, $m = 1, 3, 5, \ldots$ odd

$R(r) = c_1 J_m(\lambda_{mn} r) + c_2 Y_m(\lambda_{mn} r)$, $\lim_{r \to 0} R(r)$ bounded $\Rightarrow c_2 = 0$.

$R(1) = c_1 J_m(\lambda_{mn} r) = 0$, so λ_{mn} satisfies $J_m(\lambda_{mn}) = 0$, $m = 1, 3, 5, \ldots$
$n = 1, 2, 3, \ldots$

Superposition Principle:

$$u(r, \theta, t) = \sum_{m \text{ odd}}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\theta) J_m(\lambda_{mn} r) e^{-k\lambda_{mn} t}$$

$$u(r, \theta, 0) = T_0 r^3 \sin(3\theta) = \sum_{m \text{ odd}}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\theta) J_m(\lambda_{mn} r)$$

Orthogonality $\Rightarrow A_{mn} = 0$ for $m \neq 3$. Thus

$$T_0 r^3 = \sum_{n=1}^{\infty} A_{3n} J_3(\lambda_{3n} r)$$

$$A_{3n} = \frac{T_0 \int_0^1 r^4 J_3(\lambda_{3n} r)\, dr}{\int_0^1 r J_3^2(\lambda_{3n} r)\, dr}$$

$$u(r, \theta, t) = \sum_{n=1}^{\infty} A_{3n} \sin(3\theta) J_3(\lambda_{3n} r) e^{-k\lambda_{3n} t}$$