Partial Differential Equations

Work from the text: **13.2:** 3, 10

1. a. Consider a one-dimensional rod that is insulated along its edges. Assume that it has a length of 10 cm. The rod is initially placed so that one end is 0°C and the other end is 100°C. It is allowed to come to a steady-state temperature distribution. Find this temperature distribution, $u_e(x)$.

b. At time $t = 0$, the one-dimensional rod from Part a is insulated on both ends. This implies that the rod satisfies the PDE:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad t > 0, \quad 0 < x < 10, \\ \text{Boundary Conditions}: &\quad \frac{\partial u(0,t)}{\partial x} = 0, \qquad \frac{\partial u(10,t)}{\partial x} = 0, \quad t > 0, \\ \text{Initial Conditions}: &\quad u(x,0) = u_e(x), \quad 0 < x < 10, \end{aligned}$$

where $u_e(x)$ is the steady state temperature distribution from Part a. Find the solution to this problem, including the Fourier coefficients. Create a graphic simulation showing the 3D plot of temperature as a function of t and x, using 20 and 200 terms (Fourier coefficients) to approximate the solution with $t \in [0, 20]$.

2. Solve Laplace's equation

$$\nabla^2 u = 0,$$

inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the boundary conditions:

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(L,y) = 0,$$

$$u(x,0) = \begin{cases} 0, & x > L/2 \\ 1, & x < L/2 \end{cases}, \qquad \frac{\partial u}{\partial y}(x,H) = 0.$$

3. Solve Laplace's equation inside a semicircle of radius a, $(0 < r < a,\ 0 < \theta < \pi)$ subject to the boundary conditions.

a. $u = 0$ on the diameter, and $u(a,\theta) = g(\theta)$.

b. The diameter is insulated, and $u(a,\theta) = g(\theta)$.

4. A better model for the string problem is given by the nonhomogeneous partial differential equation:

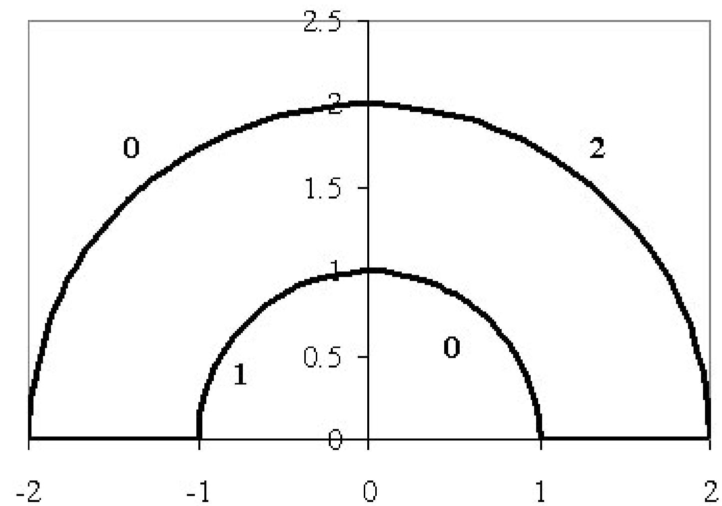
$$u_{tt} + 0.2\, u_t = u_{xx}.$$

Assume that the ends of the string are fixed with $u(0,t) = 0$ and $u(1,t) = 0$.

Suppose that the initial displacement satisfies $u(x,0) = 0$ and the initial velocity is 1 at each point of the string, *i.e.*, $u_t(x,0) = 1$. Find $u(x,t)$ and determine the limit of $u(x,t)$ as $t \to \infty$.

5. Find the steady-state temperature distribution for the Figure below (assuming the faces are insulated). The region is a semi-annular region satisfying Laplace's equation, where the edges along the x-axis are insulated. Along the semi-circular edges, we have:

$$u(1,\theta) = \begin{cases} 0, & 0 \le \theta < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} \le \theta \le \pi \end{cases}, \qquad u(2,\theta) = \begin{cases} 2, & 0 \le \theta < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le \theta \le \pi \end{cases}.$$



6. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial^2 x}$$

with $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(2,t) = -0.5\,u(2,t)$, and $u(x,0) = f(x)$.

a. Determine the appropriate Sturm-Liouville problem, solve it, and numerically find the first 5 eigenvalues.

b. Show that the eigenfunctions are orthogonal.

c. Write the solution to the partial differential equation.