Homework 6 - Approximation Theory - Solutions

BF 8.1.6. a. The data set for this problem is

x_i	0.2	0.3	0.6	0.9	1.1	1.3	1.4	1.6
y_i	0.050446	0.098426	0.33277	0.7266	1.0972	1.5697	1.8487	2.5015

The discrete least squares linear fit for the data is found and is given by

$$P_1(x) = 1.66554x - 0.512457.$$

The least sum of square errors for the linear model is $SSE = 0.335590$. The eigenvalues for the matrix A^TA are $\lambda_1 = 15.7688$ and $\lambda_2 = 0.95125$, which gives a condition number of 16.5769. For the linear model we have $BIC = -21.2116$ and $AIC = 1.33256$.

b. The discrete least squares quadratic fit for the data is found and is given by

$$P_2(x) = 1.12942x^2 - 0.311403x + 0.085144.$$

The least sum of square errors for the quadratic model is $SSE = 0.0024199$. The eigenvalues for the matrix A^TA are $\lambda_1 = 29.91076$, $\lambda_2 = 2.262135$, and $\lambda_3 = 0.057902$, which gives a condition number of 516.571. For the quadratic model we have $BIC = -58.5894$ and $AIC = -36.12470$.

c. The discrete least squares cubic fit for the data is found and is given by

$$P_3(x) = 0.266208x^3 + 0.402932x^2 + 0.248386x - 0.018401.$$

The least sum of square errors for the cubic model is $SSE = 5.07467 \times 10^{-6}$. The eigenvalues for the matrix A^TA are $\lambda_1 = 59.7619$, $\lambda_2 = 3.71247$, $\lambda_3 = 0.237906$, and $\lambda_4 = 0.0025293$ which gives a condition number of 2.3628×10^4. For the cubic model we have $BIC = -105.848$ and $AIC = -83.4625$.

The AIC and BIC suggest that the cubic model is better.

d. The discrete least squares exponential fit for the data is found and is given by

$$y_e(x) = 0.0457075e^{2.70729x}.$$

The least sum of square errors for the exponential model is $SSE = 1.07505$, which is worse than the linear fit, a model that also has two parameters.

e. The discrete least squares power law fit for the data is found and is given by

$$y_p(x) = 0.950156x^{1.87201}.$$

The least sum of square errors for the power law model is $SSE = 0.0544768$, which is better than the linear fit, a model that also has two parameters.

Problem 1: The data set for the population of the U. S. is given below;

Year	Census	Year	Census	Year	Census
1790	3.93	1870	39.82	1950	151.33
1800	5.31	1880	50.16	1960	179.32
1810	7.24	1890	62.95	1970	203.30
1820	9.64	1900	75.99	1980	226.55
1830	12.87	1910	91.97	1990	248.71
1840	17.07	1920	105.71	2000	281.42
1850	23.19	1930	122.78		
1860	31.43	1940	131.67		

We set $t = 0$ for 1790 (except for the Power Law model), then find the best linear, quadratic, cubic, exponential, and power law models.

a. The discrete least squares linear fit for the data is found and is given by

$$P_1(t) = 1.28794t - 40.5812.$$

The least sum of square errors for the linear model is $SSE = 12,834$. The eigenvalues for the matrix A^TA are $\lambda_1 = 3.3112 \times 10^5$ and $\lambda_2 = 5.8834$, which gives a condition number of 5.6279×10^4. For the linear model we have $BIC = 146.296$ and $AIC = 206.547$.

b. The discrete least squares quadratic fit for the data is found and is given by

$$P_2(t) = 0.00668498t^2 - 0.115903t + 6.21364.$$

The least sum of square errors for the quadratic model is $SSE = 171.107$. The eigenvalues for the matrix A^TA are $\lambda_1 = 9.1718 \times 10^9$, $\lambda_2 = 2.0644 \times 10^4$, and $\lambda_3 = 2.9154$, which gives a condition number of 3.14596×10^9. For the quadratic model we have $BIC = 54.401$ and $AIC = 113.561$.

c. The discrete least squares cubic fit for the data is found and is given by

$$P_3(t) = 0.00000615970t^3 + 0.00474468t^2 + 0.0433247t + 3.75592.$$

The least sum of square errors for the cubic model is $SSE = 138.278$. The eigenvalues for the matrix A^TA are $\lambda_1 = 3.0223 \times 10^{14}$, $\lambda_2 = 2.5320 \times 10^8$, $\lambda_3 = 3293$, and $\lambda_4 = 2$ which gives a condition number of 1.5943×10^{14}, suggesting problems with accuracy. For the cubic model we have $BIC = 52.805$ and $AIC = 110.874$.

d. The discrete least squares exponential fit for the data is found and is given by

$$P_e(t) = 6.07820e^{0.0201782t}.$$

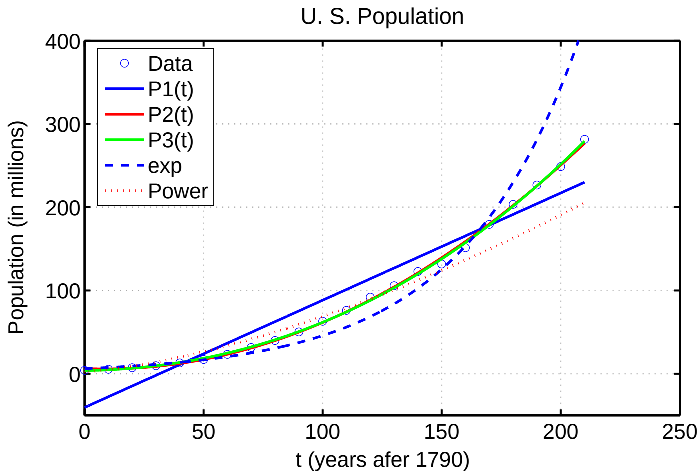
The least sum of square errors for the exponential model is $SSE = 34,730$, which is worse than the linear fit, a model that also has two parameters. For this model we have $BIC = 168.197$ and $AIC = 228.448$.

e. To fit the power law, we must shift that data so that the initial value is not $t = 0$. We shifted the data to $t = 10$ for 1790 to obtain the following discrete least squares power law fit

$$P_p(t) = 0.0415061(t + 10)^{1.57692},$$

where this t agrees with the models above. The least sum of square errors for the power law model is $SSE = 15,278$, which is slightly worse than the linear fit, a model that also has two parameters. For this model we have $BIC = 150.130$ and $AIC = 210.381$.

The AIC and BIC information criteria show that the quadratic and cubic models are just about equally good and superior to all the two parameter models.



BF 8.2.2. a. The linear least squares polynomial approximation on the interval $[-1, 1]$, $P_1(x) = a_1 x + a_0$.

$$a_0 \int_{-1}^{1} 1\,dx + a_1 \int_{-1}^{1} x\,dx = \int_{-1}^{1} (x^2 - 2x + 3)\,dx \qquad 2a_0 + 0 \cdot a_1 = \tfrac{20}{3},$$
$$a_0 \int_{-1}^{1} x\,dx + a_1 \int_{-1}^{1} x^2\,dx = \int_{-1}^{1} x(x^2 - 2x + 3)\,dx \qquad 0 \cdot a_0 + \tfrac{2}{3} a_1 = -\tfrac{4}{3}.$$

Thus, $a_0 = 10/3$ and $a_1 = -2$, so the least squares best fit polynomial is $P_1(x) = -2x + \frac{10}{3}$.

e. The linear least squares polynomial approximation on the interval $[-1, 1]$, $P_1(x) = a_1 x + a_0$.

$$a_0 \int_{-1}^{1} 1\,dx + a_1 \int_{-1}^{1} x\,dx = \int_{-1}^{1} \left(\frac{1}{2}\cos(x) + \frac{1}{3}\sin(2x)\right) dx \qquad 2a_0 + 0 \cdot a_1 = 0.841471,$$
$$a_0 \int_{-1}^{1} x\,dx + a_1 \int_{-1}^{1} x^2\,dx = \int_{-1}^{1} x \left(\frac{1}{2}\cos(x) + \frac{1}{3}\sin(2x)\right) dx \qquad 0 \cdot a_0 + \tfrac{2}{3} a_1 = 0.290265.$$

Thus, $a_0 = 0.420735$ and $a_1 = 0.435398$, so the least squares best fit polynomial is $P_1(x) = 0.435398\,x + 0.420735$.

BF 8.2.4. a. The linear least squares polynomial approximation on the interval $[-1, 1]$, $P_2(x) = a_2 x^2 + a_1 x + a_0$.

$$a_0 \int_{-1}^{1} 1\,dx + a_1 \int_{-1}^{1} x\,dx + a_2 \int_{-1}^{1} x^2\,dx = \int_{-1}^{1} (x^2 - 2x + 3)\,dx,$$

$$
\begin{aligned}
2a_0 + 0 \cdot a_1 + \frac{2}{3}a_2 &= \frac{20}{3}, \\
a_0 \int_{-1}^{1} x\,dx + a_1 \int_{-1}^{1} x^2\,dx + a_2 \int_{-1}^{1} x^3\,dx &= \int_{-1}^{1} x(x^2 - 2x + 3)\,dx, \\
0 \cdot a_0 + \frac{2}{3}a_1 + 0 \cdot a_2 &= -\frac{4}{3}, \\
a_0 \int_{-1}^{1} x^2\,dx + a_1 \int_{-1}^{1} x^3\,dx + a_2 \int_{-1}^{1} x^4\,dx &= \int_{-1}^{1} x^2(x^2 - 2x + 3)\,dx, \\
\frac{2}{3}a_0 + 0 \cdot a_1 + \frac{2}{5}a_2 &= \frac{12}{5},
\end{aligned}
$$

It is not surprising that the solution is $a_0 = 3$, $a_1 = -2$, and $a_2 = 1$, so the least squares best fit polynomial is $P_2(x) = x^2 - 2x + 3$.

e. The linear least squares polynomial approximation on the interval $[-1, 1]$, $P_2(x) = a_2x^2 + a_1x + a_0$.

$$
\begin{aligned}
a_0 \int_{-1}^{1} 1\,dx + a_1 \int_{-1}^{1} x\,dx + a_2 \int_{-1}^{1} x^2\,dx &= \int_{-1}^{1} \left(\frac{1}{2}\cos(x) + \frac{1}{3}\sin(2x)\right)\,dx, \\
2a_0 + 0 \cdot a_1 + \frac{2}{3}a_2 &= 0.841471, \\
a_0 \int_{-1}^{1} x\,dx + a_1 \int_{-1}^{1} x^2\,dx + a_2 \int_{-1}^{1} x^3\,dx &= \int_{-1}^{1} x\left(\frac{1}{2}\cos(x) + \frac{1}{3}\sin(2x)\right)\,dx, \\
0 \cdot a_0 + \frac{2}{3}a_1 + 0 \cdot a_2 &= 0.290265, \\
a_0 \int_{-1}^{1} x^2\,dx + a_1 \int_{-1}^{1} x^3\,dx + a_2 \int_{-1}^{1} x^4\,dx &= \int_{-1}^{1} x^2\left(\frac{1}{2}\cos(x) + \frac{1}{3}\sin(2x)\right)\,dx, \\
\frac{2}{3}a_0 + 0 \cdot a_1 + \frac{2}{5}a_2 &= 0.239134,
\end{aligned}
$$

From the middle equation, $a_1 = 0.435398$. Simultaneously, solving the first and last equations gives $a_0 = 0.498279$ and $a_2 = -0.232631$, so the least squares best fit polynomial is $P_2(x) = -0.232631\,x^2 + 0.435398\,x + 0.498279$.

BF 8.2.6. a. The least squares error is given by:

$$
E = \int_{-1}^{1} ((x^2 - 2x + 3) - (x^2 - 2x + 3))^2 dx = 0.
$$

BF 8.2.6. e. The least squares error is given by:

$$E = \int_{-1}^{1} \left(\left(\frac{1}{2}\cos(x) + \frac{1}{3}\sin(2x) \right) - (-0.232631\, x^2 + 0.435398\, x + 0.498279) \right)^2 dx = 0.00575719.$$

BF 8.2.7. b. The Gram-Schmidt process gives the following orthogonal polynomials (with the help of Maple). Let $\phi_0(x) = 1$ and $\phi_1(x) = x - b_1$, then

$$b_1 = \frac{\int_0^2 x\, dx}{\int_0^2 1\, dx} = \frac{2}{2} = 1,$$

so $\phi_1(x) = x - 1$. Next

$$b_2 = \frac{\int_0^2 x(\phi_1(x))^2\, dx}{\int_0^2 (\phi_1(x))^2\, dx} = \frac{\int_0^2 x(x-1)^2\, dx}{\int_0^2 (x-1)^2\, dx} = 1,$$

and

$$c_2 = \frac{\int_0^2 x\phi_1(x)\phi_0(x)\, dx}{\int_0^2 (\phi_0(x))^2\, dx} = \frac{\int_0^2 x(x-1)\, dx}{\int_0^2 1, dx} = \frac{1}{3},$$

so $\phi_2(x) = (x-1)\phi_1(x) - \frac{1}{3}\phi_0(x) = x^2 - 2x + \frac{2}{3}$. Next

$$b_3 = \frac{\int_0^2 x(\phi_2(x))^2\, dx}{\int_0^2 (\phi_2(x))^2\, dx} = \frac{\int_0^2 x(x^2 - 2x + \frac{2}{3})^2\, dx}{\int_0^2 (x^2 - 2x + \frac{2}{3})^2\, dx} = 1,$$

and

$$c_3 = \frac{\int_0^2 x\phi_2(x)\phi_1(x)\, dx}{\int_0^2 (\phi_1(x))^2\, dx} = \frac{\int_0^2 x(x-1)(x^2 - 2x + \frac{2}{3})\, dx}{\int_0^2 (x-1)^2, dx} = \frac{4}{15},$$

so $\phi_3(x) = (x-1)\phi_2(x) - \frac{4}{15}\phi_1(x) = x^3 - 3\,x^2 + \frac{12}{5}\,x - \frac{2}{5}$.

BF 8.2.8. d. By orthogonality, the coefficients are easily found. We have

$$a_0 = \frac{\int_0^2 e^x\phi_0(x)dx}{\int_0^2 (\phi_0(x))^2 dx} = \frac{\int_0^2 e^x dx}{\int_0^2 1\, dx} = \frac{e^2 - 1}{2} = 3.194528,$$

and

$$a_1 = \frac{\int_0^2 e^x\phi_1(x)dx}{\int_0^2 (\phi_1(x))^2 dx} = \frac{\int_0^2 e^x(x-1)dx}{\int_0^2 (x-1)^2 dx} = 3,$$

so $P_1(x) = 3.194528\phi_0(x) + 3\phi_1(x) = 3x + 0.194528$.

BF 8.2.10. d. With the results of 8.2.6 and orthogonality, the polynomial $P_2(x)$ is easily found. We have

$$a_2 = \frac{\int_0^2 e^x\phi_(2x)dx}{\int_0^2 (\phi_2(x))^2 dx} = \frac{\int_0^2 e^x(x^2 - 2x + \frac{2}{3})dx}{\int_0^2 (x^2 - 2x + \frac{2}{3})^2 dx} = 1.45896,$$

so $P_2(x) = 3.194528\phi_0(x) + 3\phi_1(x) + 1.45896\phi_2(x) = 1.167168 + 0.08208\,x + 1.45896\,x^2$.

BF 8.3.2. b. The zeroes of $T_4(x)$ occur at

$$\bar{x}_1 = \cos(\pi/8) = 0.9238795 \qquad \bar{x}_2 = \cos(3\pi/8) = .3826834$$
$$\bar{x}_4 = \cos(7\pi/8) = -\bar{x}_1 \qquad \bar{x}_3 = \cos(5\pi/8) = -\bar{x}_2$$

The corresponding function values at those points are

$$\sin(x_1) = 0.7979459 \qquad \sin(x_2) = 0.3734111$$
$$\sin(x_4) = -0.7979459 \qquad \sin(x_3) = -0.3734111$$

The Lagrange polynomial is given by

$$\begin{aligned} P(x) \;=\; & 0.6107214\ (x-0.3826834)\,(x+0.3826834)\,(x+0.9238795) \\ & -0.6899738\ (x-0.9238795)\,(x+0.3826834)\,(x+0.9238795) \\ & -0.6899738\ (x-0.9238795)\,(x-0.3826834)\,(x+0.9238795) \\ & +0.6107214\ (x-0.9238795)\,(x-0.3826834)\,(x+0.3826834) \end{aligned}$$

which simplifies to

$$P(x) = -0.1585049\,x^3 + 0.99898289\,x.$$

BF 8.3.4. b. A bound on the error is given by

$$\max_{x\in[-1,1]} |\sin(x) - P_3(x)| \le \frac{1}{2^3 4!} \max_{x\in[-1,1]} |f^{(4)}(x)| = \frac{\sin(1)}{192} = 0.0043827.$$

BF 8.4.2. The Taylor (Maclaurin) series for $f(x) = x\ \ln(x+1)$ to order 3 is given by $f(x) = x^2 - \frac{1}{2}x^3 + O\left(x^4\right)$, so the Padé approximation with $n = 3$ and $m = 0$ is given by:

$$r_{3,0}(x) = x^2 - \frac{1}{2}x^3.$$

For $n = 2$ and $m = 1$, we examine

$$\left(x^2 - \frac{1}{2}x + ...\right)(1 + q_1 x) - (p_0 + p_1 x + p_2 x^2) = -p_0 - p_1 x + (1 - p_2)x^2 + \left(q_1 - \frac{1}{2}\right)x^3,$$

so $p_0 = p_1 = 0$, $p_2 = 1$, and $q_1 = 1/2$, which gives the Padé approximation:

$$r_{2,1}(x) = \frac{x^2}{1 + \frac{1}{2}x}.$$

Because the Taylor series begins with x^2, the other two Padé approximations with $n = 1$ and $m = 2$, $r_{1,2}(x)$, and $n = 0$ and $m = 3$, $r_{0,3}(x)$, cannot be generated. Below is a table showing the function values and the Padé approximations for several x values.

$0.2 \times i$	$x\ \ln(x+1)$	$r_{3,0}(x)$	$r_{2,1}(x)$
0.2	0.036464311	0.036	0.036363636
0.4	0.134588895	0.128	0.133333333
0.6	0.282002178	0.252	0.276923077
0.8	0.470229332	0.384	0.457142857
1.0	0.693147181	0.5	0.666666667