

**Homework 4 - Solutions**

**WeBWorK** scores are multiplied by 1.5, as they were under weighted in the program.

**WeBWorK Problem 2. c.** We assume the function is given by (function will vary will different versions):

$$f(x) = x^2 \cos(x) - 3x,$$

which has  $f(0.18) = -.5081234644$ . For  $n = 1$ , Theorem 3.3 gives the error bound

$$|f(x) - P_1(x)| = \left| \frac{f''(\xi(x))}{2!} (x - 0.1)(x - 0.2) \right|$$

where

$$f''(x) = (2 - x^2) \cos(x) - 4x \sin(x).$$

On the interval  $x \in [0.1, 0.2]$ , cosine is decreasing and sine is increasing, so it is easy to see that  $f''(x)$  is at a maximum at  $x = 0.1$ . It follows that

$$|f''(x)| \leq f''(0.1) = 1.99 \cos(0.1) - 0.4 \sin(0.1) < 1.95.$$

Thus,

$$|f(0.18) - P_1(0.18)| < \left| \frac{1.95}{2!} (0.18 - 0.1)(0.18 - 0.2) \right| = 0.00156.$$

The absolute error is

$$|f(0.18) - P_1(0.18)| < |-.5081234644 + 0.506647844| = 0.0014756204.$$

For  $n = 2$ , Theorem 3.3 gives the error bound

$$|f(x) - P_2(x)| = \left| \frac{f^{(3)}(\xi(x))}{3!} (x - 0.1)(x - 0.2)(x - 0.3) \right|$$

where

$$f^{(3)}(x) = (x^2 - 6) \sin(x) - 6x \cos(x).$$

This third derivative is negative on the interval  $x \in [0.1, 0.3]$  with cosine decreasing and sine increasing. We can obtain a bound on the third derivative by evaluating:

$$|f^{(3)}(x)| \leq |(0.1^2 - 6) \sin(0.3) - 6(0.3) \cos(0.1)| < 3.57.$$

Thus,

$$|f(0.18) - P_2(0.18)| < \left| \frac{3.57}{3!} (0.18 - 0.1)(0.18 - 0.2)(0.18 - 0.3) \right| = 0.00011424.$$

The absolute error is

$$|f(0.18) - P_2(0.18)| < |-.5081234644 + 0.508049852| = 0.0000736124.$$

**WeBWorK 4. c.** Since the population of the U. S. is given from 1940 to 1990, then the best approximation is the interpolatory value at 1965. The worst values will be beyond the range of the data at 1930 and 2010. We expect the value at 1965 to be reasonably accurate, while using an interpolatory polynomial is unlikely to give good values for either 1930 or 2010. We see the predicted population in 1930 is larger than the 1940 population and the population in 2010 is smaller than the 1990 population, which is nonsense.

**Problem 1:** a. The data for the number of hare pelts (in thousands) in years after 1900 is given in the table below.

Year	0	1	2	3	4	5	6	7	8	9	10
Pelts	30	47.2	70.2	77.4	36.3	20.6	18.1	21.4	22	25.4	27.1

The Lagrange polynomial interpolation from Maple is given by

$$\begin{aligned}
 P(t) = & \frac{1}{120960} (t-1)(t-2)(t-3)(t-4)(t-5)(t-6)(t-7)(t-8)(t-9)(t-10) \\
 & -0.0001300705467 t(t-2)(t-3)(t-4)(t-5)(t-6)(t-7)(t-8)(t-9)(t-10) \\
 & +0.0008705357143 t(t-1)(t-3)(t-4)(t-5)(t-6)(t-7)(t-8)(t-9)(t-10) \\
 & -0.002559523810 t(t-1)(t-2)(t-4)(t-5)(t-6)(t-7)(t-8)(t-9)(t-10) \\
 & +0.002100694444 t(t-1)(t-2)(t-3)(t-5)(t-6)(t-7)(t-8)(t-9)(t-10) \\
 & -0.001430555556 t(t-1)(t-2)(t-3)(t-4)(t-6)(t-7)(t-8)(t-9)(t-10) \\
 & +0.001047453704 t(t-1)(t-2)(t-3)(t-4)(t-5)(t-7)(t-8)(t-9)(t-10) \\
 & -0.0007076719577 t(t-1)(t-2)(t-3)(t-4)(t-5)(t-6)(t-8)(t-9)(t-10) \\
 & +\frac{11}{40320} t(t-1)(t-2)(t-3)(t-4)(t-5)(t-6)(t-7)(t-9)(t-10) \\
 & -0.00006999559083 t(t-1)(t-2)(t-3)(t-4)(t-5)(t-6)(t-7)(t-8)(t-10) \\
 & +0.000007468033510 t(t-1)(t-2)(t-3)(t-4)(t-5)(t-6)(t-7)(t-8)(t-9)
 \end{aligned}$$

This is simplified with the Maple expand command to give

$$\begin{aligned}
 P(t) = & 30 + 668.2403186 t - 1729.502638 t^2 + 1807.604284 t^3 - 995.7865571 t^4 \\
 & + 325.0923472 t^5 - 66.37664543 t^6 + 8.582431400 t^7 - 0.6835722565 t^8 \\
 & + 0.03062265770 t^9 - 0.0005905809097 t^{10}
 \end{aligned}$$

A graph of this polynomial and the data are shown below.

b. We differentiate the polynomial above (using Maple) and set the derivative equal to zero to find the maximum and minimum. We use the graph to visualize where the maximum and minimum occur. The maximum occurs at  $t_{max} = 0.32188$  with the maximum value  $P(t_{max}) = 116.550$ . Like the data, the interpolation shows a wide minimum with the absolute minimum for  $t \in [0, 10]$  being  $t_{min} = 8.57148$  with the minimum value of  $P(t_{min}) = 16.911$ . Given the data, we see that these extrema are not where the data suggests the extrema should occur, but rather from the high degree of oscillation by interpolation polynomials that can occur near the end points. The maximum and minimum are found by a secant method near the extrema.

c. The Lagrange polynomial interpolation from Maple using just the first 5 years of data is given by

$$P(t) = -1/4 (t-1)(t-2)(t-3)(t-4)(t-5) + 1.966667 t(t-2)(t-3)(t-4)(t-5)$$



$$\begin{aligned}
 & -5.85t(t-1)(t-3)(t-4)(t-5) + 6.45t(t-1)(t-2)(t-4)(t-5) \\
 & -1.5125t(t-1)(t-2)(t-3)(t-5) + 0.1716666667t(t-1)(t-2)(t-3)(t-4),
 \end{aligned}$$

which simplifies to the polynomial

$$P(t) = 30 + 33.245t - 40.0875t^2 + 33.2791667t^3 - 10.2125t^4 + 0.975833t^5.$$

A graph of this polynomial and the data are shown below.



By differentiating, we find the maximum occurs at  $t_{max} = 2.6967$  with  $P(t_{max}) = 79.845$ . Similarly, we find the minimum occurs at  $t_{max} = 4.6912$  with  $P(t_{max}) = 10.5056$ . We see that these values are more fitting with the data though the minimum is a little early and a bit low because of the oscillation error noted above.

d. The primary strength of modeling the data using Lagrange interpolation polynomials is the relative ease by which this technique can be employed. Also, it is very easy to work with the polynomial data to find extrema as was shown above. It provides a reasonable first approach to smoothing the data.

The weaknesses outweigh the strengths in modeling. The interpolation polynomials do a poor job of connecting the data, especially near the end points. The data suggest an oscillatory model, which would be better accomplished with trig functions instead of polynomials. Finally, a data fitting algorithm gives no insight into the dynamics of the populations and why it might be oscillating.