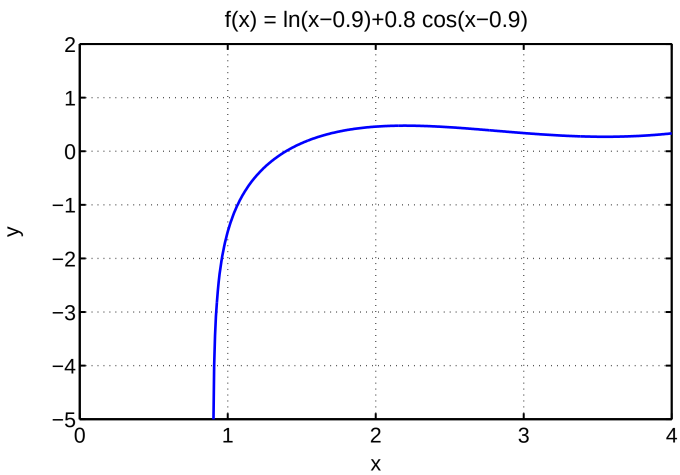
Homework 2 - Solutions

WeBWorK 2 d. The bisection method converges independent of the function. Thus, it is a very stable method, but it converges slowly. The secant method converges at a super-linear rate. For this example, it converges almost as rapidly as Newton's method without the cost of finding the derivative. Newton's method converges quadratically provided the initial guess is sufficiently close.

WeBWorK 3 d. The function that is illustrated is given by:

$$f(x) = \ln(x - 0.9) + 0.8\cos(x - 0.9).$$

Below is a graph of the function with the zero of the function clearly visible near $x_e \approx 1.4$. There is a vertical asymptote at $x = 0.9$ with the function undefined for values of $x \leq 0.9$. Newton's method should converge for initial values $x_0 \in (0.9, x_e)$ as the tangent lines converge toward the zero of the function. In the other direction we see that the derivative is zero near $x = 2$, so Newton's method would fail. In fact, it will fail whenever the tangent line intersects at some value $x_1 < 0.9$, so the interval for Newton's method to converge is limited to some interval less than the interval $(0.9, 2)$.

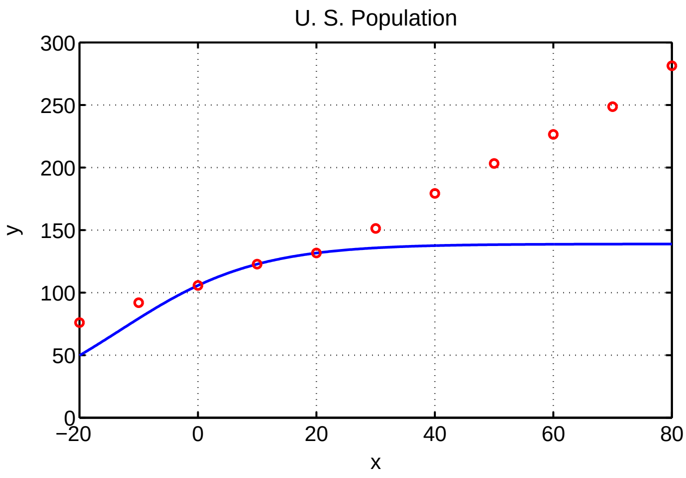


WeBWorK 4 c. The best fitting model (for one version) is given by

$$P(t) = \frac{138.907}{1 + 0.31403e^{-0.087129t}},$$

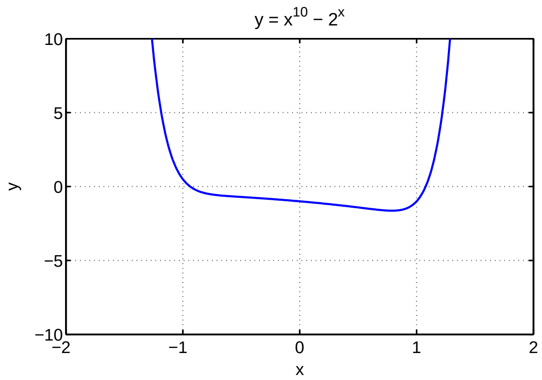
which with the population (in thousands) and $t = 0$ in 1920. The graph fits the years 1920, 1930, and 1940 exactly, but does poorly at predicting any other population in the twentieth century, largely because of the low growth rate in the 1930s.

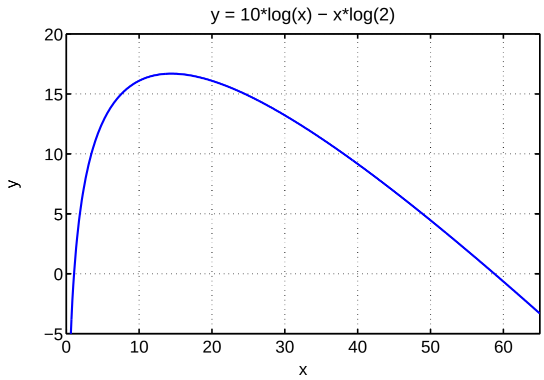
The solution of this equation is easily found by first eliminating the linear variables P_L and c. The result is a nonlinear equation in the unknown k, which is readily solved using Newton's method or the secant method. Once we find k, the other variables are easily obtained by substitution.



1. For each of the equations below, we used MatLab to graph the functions, then applied Newton's method to find the roots of the equation with starting values near the ones observed on the graphs.

a. To solve $x^{10} = 2^x$, we write $f(x) = x^{10} - 2^x$, which is easily solved by Newton's method for the three roots. Graphically, it is easier to visualize the logarithm of the function or $f(x) = 10\ln(x) - x\ln(2)$.





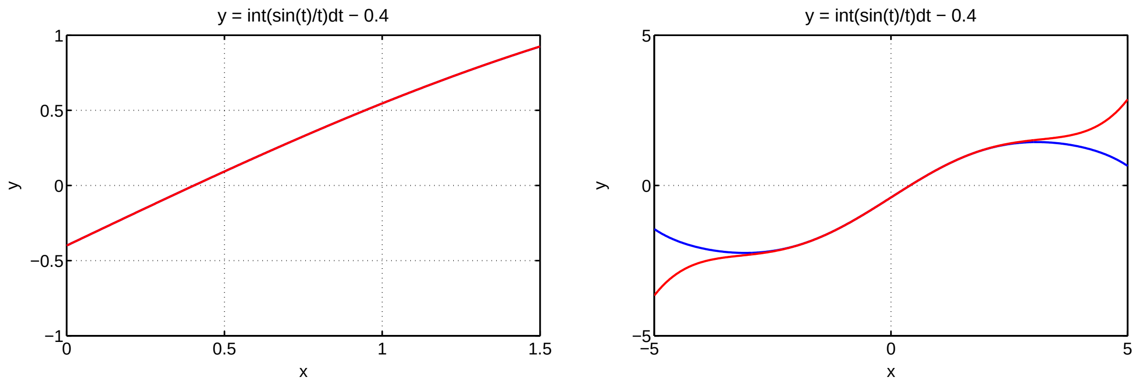
There are three roots at

$$
\begin{aligned}
x_1 &= -0.937109, \\
x_2 &= 1.07755, \\
x_3 &= 58.7701,
\end{aligned}
$$

b. To solve $0.4 = \int_0^x \frac{\sin(t)}{t}\,dt$, we take $\sin(t)/t$ and expand it as a Maclaurin series, then integrating term by term, we obtain

$$\int_0^x \frac{\sin(t)}{t}\,dt = x - \frac{1}{18}\,x^3 + \frac{1}{600}\,x^5 - \frac{1}{35280}\,x^7 + O\left(x^9\right).$$

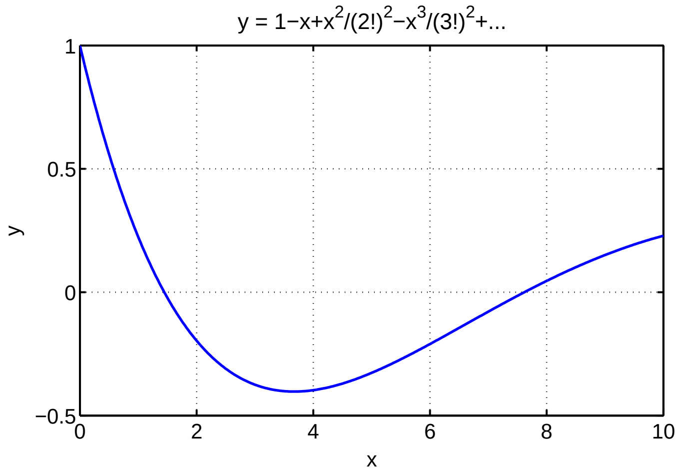
Below are graphs of the approximating polynomials of order 5 and 7 (minus 0.4).



The polynomial part of this expression minus 0.4 is inserted into Newton's method to give the root $x = 0.4036356$. (The polynomial of order 5 is sufficient to give the root to 5 significant figures, giving $x = 0.4036355$.) The figures above show the 7^{th} order polynomial approximation in blue and 5^{th} order polynomial approximation in red.

c. We expand the series to as many terms as necessary to get the two smallest positive roots of

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - ... = 0.$$



The smallest two positive roots are

$$\begin{aligned} x_1 &= 1.445796, \\ x_2 &= 7.6178156. \end{aligned}$$

The second root requires a 10^{th} order polynomial approximating the series to get 5 significant figures, while a 12^{th} order polynomial approximation gives the accuracy stated above.

2. **Radius of the Earth**: Let R be the radius of the Earth in miles. Define in polar coordinates each of the places beginning with Mission Valley at $(r, \theta) = (R + 40/5280, 0)$. Cowles Mountain has coordinates $(R + 1591/5280, 9.2/R)$, while Mount Cuyumaca has coordinates $(R + 6512/5280, 34.85/R)$. In Cartesian coordinates, the point in Mission Valley is $(x, y) = (R + 40/5280, 0)$. For Cowles Mountain $(x, y) = ((R + 1591/5280)\cos(9.2/R), (R + 1591/5280)\sin(9.2/R))$, while for Mount Cuyumaca $(x, y) = ((R+6512/5280)\cos(34.85/R), (R+6512/5280)\sin(34.85/R))$. Since the points line up,

$$\frac{\Delta y}{\Delta x} = \frac{\left(R + \frac{1591}{5280}\right)\sin\left(\frac{9.2}{R}\right)}{\left(R + \frac{1591}{5280}\right)\cos\left(\frac{9.2}{R}\right) - \left(R + \frac{40}{5280}\right)} = \frac{\left(R + \frac{6512}{5280}\right)\sin\left(\frac{34.85}{R}\right)}{\left(R + \frac{6512}{5280}\right)\cos\left(\frac{34.85}{R}\right) - \left(R + \frac{40}{5280}\right)}$$

Since this expression is fairly complicated, it is easiest to use the secant method to find the solution, R, for the equation above. The result gives the radius of the Earth as 3964.55 miles.