I, ____________________________, pledge that this exam is completely my own work, and that I did not take, borrow or steal any portions from any other person. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

____
Signature

Be sure to show all your work or include a copy of your program.

1. a. This semester we learned about Gaussian quadrature. Below is a MatLab code that could be used to evaluate one of the homework problems (hyperlink available on webpage). Begin this problem by describing in detail what this MatLab code does and how it solves integrals.

```
f  = inline('sin(x^4)');
a = 0;
b = 5;
tol      = 10^(-5);
MAX_ITER = 12;
r1 = 0.8611366116;
r2 = 0.3399810436;
c1 = 0.3478548451;
c2 = 0.6521451549;
k = 1; i = 1;
h = b-a;
u1 = a+h*(1-r1)/2;
u2 = a+h*(1-r2)/2;
u3 = a+h*(1+r2)/2;
u4 = a+h*(1+r1)/2;
s = c1*(f(u1) + f(u4)) + c2*(f(u2) + f(u3));
sum = s*h/2;
fprintf('---------------------------------\n');
fprintf('  k         sum       rel_err\n');
fprintf('---------------------------------\n');
fprintf('%3d   %8.6f   %8.8f\n',k, sum, sum);
while( i < MAX_ITER )
    k = 2*k;
    s = 0;
    h = (b-a)/k;
    u1 = a+h*(1-r1)/2;
    u2 = a+h*(1-r2)/2;
```

```
    u3 = a+h*(1+r2)/2;
    u4 = a+h*(1+r1)/2;
    for j=1:k,
        s = s + c1*(f(u1) + f(u4)) + c2*(f(u2) + f(u3));
        u1 = u1 + h;
        u2 = u2 + h;
        u3 = u3 + h;
        u4 = u4 + h;
    end
    sum_k = s*h/2;
    d_int = abs(sum - sum_k);
    if( d_int<tol )
        fprintf('%3d   %8.6f   %8.8f\n',k, sum_k, d_int);
        return;
    end;
    sum = sum_k;
    fprintf('%3d   %8.6f   %8.8f\n',k, sum, d_int);
    i = i+1;
end
fprintf('WARNING: Gauss Integration did not converge after %d steps.\n',k);
```

b. There is an integral formula that is useful for evaluating Bessel functions, and it is given by the following:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin(\theta))\,d\theta \qquad n = 0, 1, ...$$

Gaussian quadrature applies to integrals that have limits from $x = -1$ to $x = 1$. Transform the above formula to one that has these limits of integration.

c. Modify the algorithm above (or write your own code) to evaluate $J_5(3)$ (accurate to 5 significant figures).

d. The 5^{th} order Bessel function, $J_5(z)$, has a zero near $z = 9$. You are given the additional identity that

$$\frac{d}{dz}J_m(z) = J_{m-1}(z) - \frac{m}{z}J_m(z).$$

With this identity, write Newton's formula with the appropriate integral forms given in Part b. Further modify the MatLab code above (or create your own) to perform a Newton's iteration where the function and its derivative are found from Gaussian quadratures. Start with $z_0 = 9$, and find the Newton iterates (allowing each quadrature to converge to at least a tolerance of 10^{-6}), until you converge to a solution that is accurate to 10^{-5}.

2. We are interested in finding the value of the following integral:

$$\int_0^{\infty} \frac{\cos(x)}{1 + x^4}dx.$$

a. Split this into two integrals (one from 0 to 1 and another from 1 to ∞). Approximate the first integral using a Gaussian quadrature with $n = 3$ and $n = 4$. Also, approximate this first integral using a Composite Simpson's rule (CSR) with $h = 0.1$. Determine a bound on the error for the CSR.

b. The remaining (transformed) integral is still singular because of the cosine function. Show that the transformed integrand tends to zero as $x \to 0$. Apply the CSR with $h = 0.1$ to this integral and determine its value.

c. Combine the CSR answers from Parts a and b to give an approximate value of the original integral. Discuss how accurate this answer is based on the approximations used.

3, a. Define the inner product between two functions, f and g, with a weighting function of $w(x) = e^{-2x}$ by:

$$\langle f, g \rangle = \int_0^\infty f(x)g(x)e^{-2x}dx.$$

Let $P_0(x) = 1$. Use the Gram-Schmidt process to find an orthogonal set of polynomials, $P_1(x) = 1 + a_1x$, $P_2(x) = 1 + a_2x + b_2x^2$, and $P_3(x) = 1 + a_3x + b_3x^2 + c_3x^3$. Create a normalized set by making

$$\bar{P}_i(x) = q_iP_i(x) \qquad \text{for some} \quad q_i, \quad i = 1, 2, 3$$

with

$$\langle \bar{P}_i, \bar{P}_i \rangle = q_i^2 \int_0^\infty P_i^2(x)e^{-2x}dx = 1.$$

b. Find the zeroes for each of these polynomials.

c. Use the polynomials, $P_i(x)$ for $i = 0, 1, 2, 3$, above to find the least squares approximating polynomial for $f(x) = e^{-x}$. Graph the least squares polynomial and $f(x)$ for $x \in [0, 5]$.

d. Let $Q(x)$ be the approximating polynomial found in Part c. The l_2 distance between the approximating polynomial and the function is given by the norm defined:

$$||Q - f||_2 = \langle (Q - f), (Q - f) \rangle^{1/2}.$$

Find the least squares distance between the approximating polynomial $Q(x)$, and the function, $f(x) = e^{-x}$.

4. Table **??** has weights for average American girls between the ages of 1 and 17.

a. Consider the data for the average weight of American girls, w, as a function of age, t. Construct discrete least squares polynomials of degrees 1, 2, 3, and 4 through the data. Give the minimum sum of square errors for each of these models. Find the eigenvalues and condition numbers for your matrices A^TA, then compare these 4 models with the Bayesian and Akaike Information Criteria.

b. We know that polynomials are unbounded, while weight tends to a maximum with aging. Thus, there are better functional relationships that model these processes. Consider a model for weight of the form

$$w(t) = \frac{W_{max}}{1 + Ae^{-rt}},$$

for some constants W_{max}, A, and r. We can rearrange this model to the following equation:

$$y(t) = \frac{1}{w(t)} - \frac{1}{W_{max}} = \frac{A}{W_{max}}e^{-rt}.$$

age(years)	weight(kg)	age(years)	weight(kg)
1	9.5	10	32.7
2	11.8	11	37.3
3	15.0	12	41.4
4	15.9	13	46.8
5	18.2	14	49.6
6	20.0	15	52.0
7	21.8	16	53.9
8	25.0	17	55.2
9	29.1		

Table 1: Weights of average American girls between ages 1 and 17.

For a given value of W_{max}, we see that this is an exponential function. Find the best fitting models for $W_{max} = 60$, 62, and 65. Determine which of these models is the best fit and compare them to the polynomial models from Part a. (Compare the sum of square errors of the models for $w(t)$, not $y(t)$.)

5. Work Burden and Faires Problem 7.3.28. Show the first 10 iterates for all three methods. State how many iterates are required for convergence to the tolerance listed in the problem (or state that the scheme diverges). Find the eigenvalues and spectral radius for the matrices of all three iterative schemes, T_j, T_g, and T_ω.

6. From Chemical Engineering, the steady state diffusion in a quiescent fluid body with a first order chemical reaction may be modeled by the following boundary value problem:

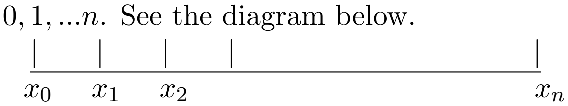
$$D\frac{d^2u}{dx^2} - Ku = 0, \qquad 0 \le x \le 1,$$

where u is the concentration of the substance, the diffusivity constant is $D = 0.01$ cm^2/sec, and the reaction coefficient is $K = 0.1$ sec^{-1}. The boundary conditions are given by:

$$u(0) = 0 \qquad \text{and} \qquad u(1) = C,$$

where the right boundary concentration is $C = 1.0$ g/cm^3.

a. Divide the interval $[0, 1]$ into n equal panels of width $h = 1/n$ with x_i being the dividing points for $i = 0, 1, ...n$. See the diagram below.



Evaluate the differential equation at each of the interior dividing points $(x_1, ..., x_{n-1})$ and approximate the derivatives by the second order formula:

$$\left.\frac{d^2u}{dx^2}\right|_{x_i} \approx \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$$

(Note that $u_0 = 0$ and $u_n = C$.) For $n = 8$ explicitly write the 7×7 linear system for the variables u_i, with $i = 1, ..., 7$.

b. This linear system is very sparse. Write the algorithm for a Jacobi iteration and a Gauss-Seidel iteration (in simplest form, *i.e.* no summations). Solve these systems to 5 significant digits, showing the concentrations at $x = 0.25$, 0.5, and 0.75. For an initial approximation, use a straight line approximation between $u(0) = 0$ and $u(1) = 1$.

c. This linear system is has a tridiagonal A, so our theorem for finding an optimal ω for the SOR method applies. Find the optimal ω and solve this linear system using the SOR method. Solve these systems to 5 significant digits, showing the concentrations at $x = 0.25$, 0.5, and 0.75. Use the same initial condition as above.

d. Create a sequence of larger systems by doubling the number of panels for each new system ($n = 16, 32, ...$). Find the approximate values of $u(x)$ at $x = 0.25$, 0.5, and 0.75 for each of your larger systems. Continue the sequence until all of the approximations at $x = 0.25$, 0.5, and 0.75 are accurate to 5 significant figures. Use the Gauss-Seidel method for solving these systems.