1. The transition matrix for red squirrels (*Sciurus vulgaris L.*), gray squirrels (*Sciurus carolinensis Gmelin*), both, or neither in that order is given by

$$T = \begin{pmatrix} 0.8797 & 0.0382 & 0.0527 & 0.0008 \\ 0.0212 & 0.8002 & 0.0041 & 0.0143 \\ 0.0981 & 0.0273 & 0.8802 & 0.0527 \\ 0.0010 & 0.1343 & 0.0630 & 0.9322 \end{pmatrix}.$$

The characteristic polynomial is given by

$$\lambda^4 - 3.4923\lambda^3 + 4.5578\lambda^2 - 2.6347\lambda + 0.56926 = 0.$$

This has the eigenvalues

$$\lambda_1 = 1, \quad \lambda_2 = 0.90828, \quad \lambda_3 = 0.81520, \quad \lambda_4 = 0.76882.$$

The dominant eigenvalue is $\lambda_1 = 1$ (as expected) with normalized eigenvector

$$\xi_1 = [0.17053, 0.055987, 0.34215, 0.43134]^T.$$

It follows that the steady state distribution of the squirrels on these 10 km square regions across Great Britain has 17.1% occupied by red squirrels, 5.6% occupied by grey squirrels, 34.2% occupied by both types of squirrel, and 43.1% occupied by neither. This model suggests that the native red squirrel is holding its own and the invasive gray squirrel will not significantly displace it.

2. a. The Leslie model is given by:

$$\begin{pmatrix} p_1(n+1) \\ p_2(n+1) \\ p_3(n+1) \\ p_4(n+1) \\ p_5(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 0.7 & 1.6 & 2.5 & 1.8 \\ 0.35 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 \end{pmatrix} \begin{pmatrix} p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \\ p_5(n) \end{pmatrix}.$$

The characteristic polynomial is

$$\lambda^5 - 0.245\lambda^3 - 0.3369\lambda^2 - 0.39375\lambda - 0.19845 = 0.$$

The eigenvalues are $\lambda_1 = 1.0475$, $\lambda_{2,3} = 0.01578 \pm 0.7254i$, and $\lambda_{4,5} = -0.5395 \pm 0.2622i$. The dominant eigenvalue (λ_1 has the normalized eigenvector $\xi_1 = [0.5701, 0.1905, 0.1091, 0.07812, 0.05220]^T$. It follows that the steady state distribution of the population is 57.01% animals 0-1 years old, 19.05% animals 1-2 years old, 10.91% animals 2-3 years old, 7.812% animals 3-4 years old, and 5.220% animals 4-5 years old. The population doubles when $\lambda_1^n = 2$ or $n = \frac{\ln(2)}{\ln(\lambda_1)} = 14.935$ years.

b. With the harvesting of 2-4 year olds, the model becomes

$$\begin{pmatrix} p_1(n+1) \\ p_2(n+1) \\ p_3(n+1) \\ p_4(n+1) \\ p_5(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 0.7 & 1.6 & 2.5 & 1.8 \\ 0.35 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.75\alpha & 0 & 0 \\ 0 & 0 & 0 & 0.7\alpha & 0 \end{pmatrix} \begin{pmatrix} p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \\ p_5(n) \end{pmatrix}.$$

The new characteristic polynomial is

$$\lambda^5 - 0.245\lambda^3 - 0.3369\lambda^2 - 0.39375\alpha\lambda - 0.19845\alpha^2 = 0.$$

For the population to remain constant, then the dominant eigenvalue has to be $\lambda_1 = 1$. It follows from the characteristic equation that

$$0.19845\alpha^2 + 0.39375\alpha - 0.419 = 0,$$

which has the solutions $\alpha = -2.7515$ and 0.767355. The negative solution doesn't make sense, so $\alpha = 0.767355$, and the model becomes:

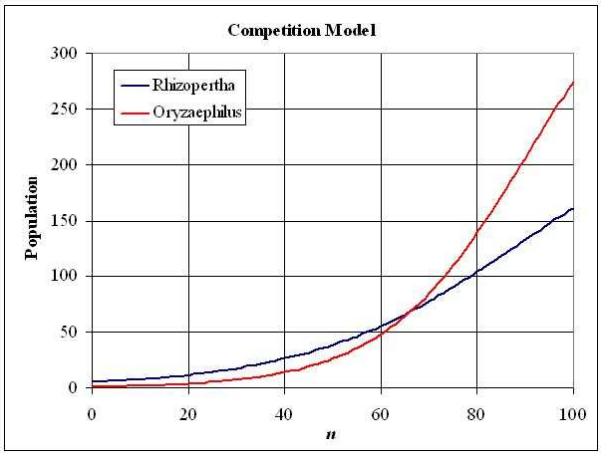
$$\begin{pmatrix} p_1(n+1) \\ p_2(n+1) \\ p_3(n+1) \\ p_4(n+1) \\ p_5(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 0.7 & 1.6 & 2.5 & 1.8 \\ 0.35 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0.57552 & 0 & 0 \\ 0 & 0 & 0 & .53715 & 0 \end{pmatrix} \begin{pmatrix} p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \\ p_5(n) \end{pmatrix},$$

which has a dominant eigenvalue $\lambda_1 = 1$ with associated eigenvector

$$\xi_1 = [0.57281, 0.20048, 0.12029, 0.069229, 0.037186]^T.$$

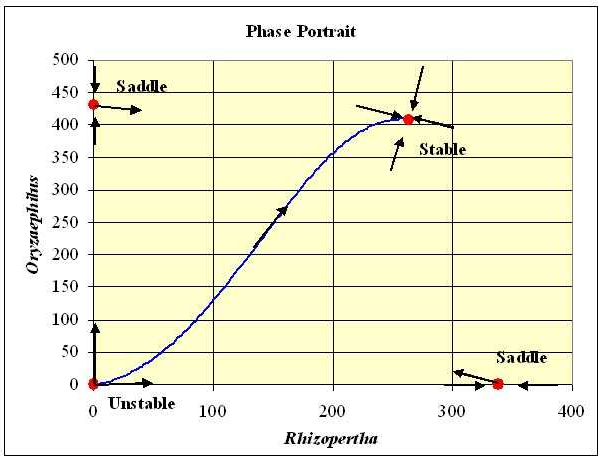
If there are 550 mature animals and they make up 3.7186% of the population, then the total population is 9412.0 animals. By age group, the population is 5391.3 animals 0-1 years old, 1887.0 animals 1-2 years old, 1132.2 animals 2-3 years old, 651.6 animals 3-4 years old, and 350 animals 4-5 years old. The normal survival of 2-3 year olds would be $0.75 \times 1132.2 = 849.15$, while the population with harvesting is 651.6, so there were 197.55 animals age 2-3 harvested. Similarly, it can be shown that 106.11 animals age 3-4 were harvested. It follows that 303.66 animals are harvested under these conditions, and the population remains steady at 9412.0 animals.

3. a. Below is a graph of the time series for the populations of *Rhizopertha dominica*, the lesser grain borer, and *Oryzaephilus surinamensis*, the saw-tooth grain beetle, competing for the same resource.



b. The equilibria for this competition model are $(R_e, O_e) = (0,0)$, $(338.5, 0)$, $(0, 431.25)$, and $(263.1, 408.2)$. The equilibrium at $(R_e, O_e) = (0,0)$ is an unstable node with eigenvalue $\lambda_1 = 1.044$ with associated eigenvector $\xi_1 = [1, 0]$ and eigenvalue $\lambda_2 = 1.069$ with associated eigenvector $\xi_2 = [0, 1]$. The equilibrium at $(R_e, O_e) = (338.5, 0)$ is a saddle node with eigenvalue $\lambda_1 = 0.956$ with associated eigenvector $\xi_1 = [1, 0]$ and eigenvalue $\lambda_2 = 1.0643$ with associated eigenvector $\xi_2 = [-0.07503, 1]$. The equilibrium at $(R_e, O_e) = (0, 431.25)$ is a saddle node with eigenvalue $\lambda_1 = 0.931$ with associated eigenvector $\xi_1 = [0, 1]$ and eigenvalue $\lambda_2 = 1.03365$ with associated eigenvector $\xi_2 = [1, -0.058816]$. Finally, the coexistence equilibrium at $(R_e, O_e) =$

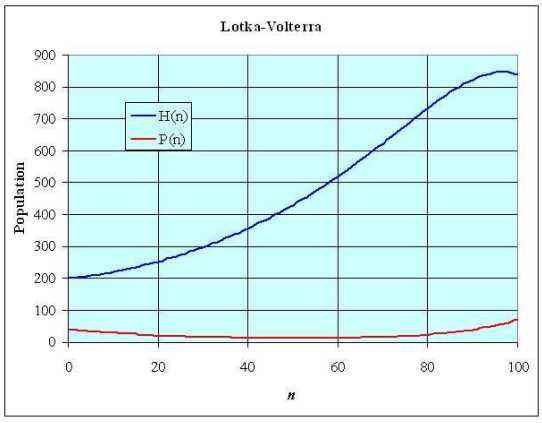
$(263.10, 408.23)$ is a stable node with eigenvalue $\lambda_1 = .9336$ with associated eigenvector $\xi_1 = [0.1923, 0.9815]$ and eigenvalue $\lambda_2 = 0.9669$ with associated eigenvector $\xi_2 = [0.9846, -0.1746]$. It follows that over a long period of time, the population should settle into the coexistence equilibrium with about 263 *Rhizopertha dominica*, and 408 *Oryzaephilus surinamensis*.



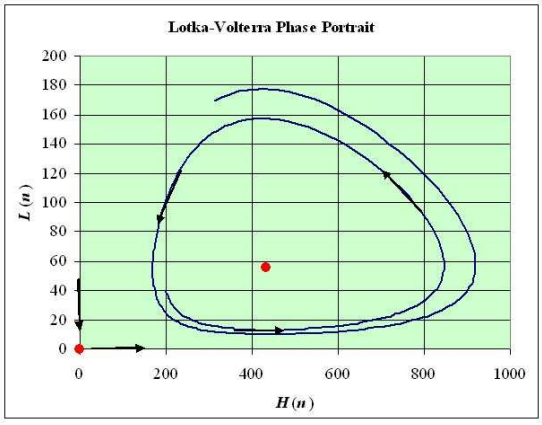
4. a. The classic Lotka-Volterra or predator-prey model given by the equation:

$$\begin{aligned} H_{n+1} &= H_n + 0.025H_n - 0.00045H_nP_n, \\ P_{n+1} &= P_n - 0.065P_n + 0.00015H_nP_n, \end{aligned}$$

where n is in days, has H_n as the prey (or host) and P_n as the predator (or parasite). Below is the simulation for 100 days of this model.



Time simulation



Phase portrait

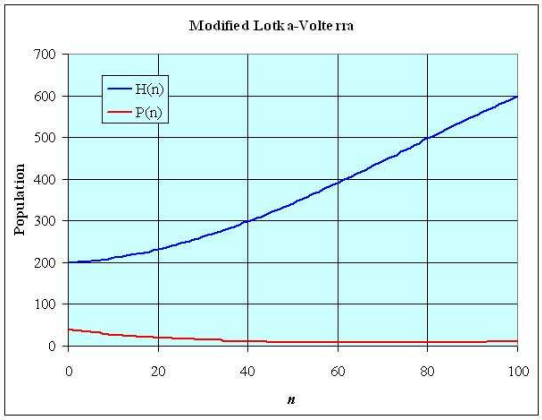
b. The equilibria for this model are $(H_e, P_e) = (0, 0)$ and $(433.3, 55.56)$. The equilibrium at $(H_e, P_e) = (0, 0)$ is a saddle node with eigenvalue $\lambda_1 = 1.025$ with associated eigenvector $\xi_1 = [1, 0]$ and eigenvalue $\lambda_2 = 0.935$ with associated eigenvector $\xi_2 = [0, 1]$. The equilibrium at $(H_e, P_e) = (433.3, 55.56)$ has eigenvalues $\lambda = 1.0000 \pm 0.040311i$, which have magnitude $|\lambda| = 1.0008$. It follows that this equilibrium produces an unstable spiral with solutions moving away from this equilibrium. See the phase portrait above.

c. The modified Lotka-Volterra model given by the equation:

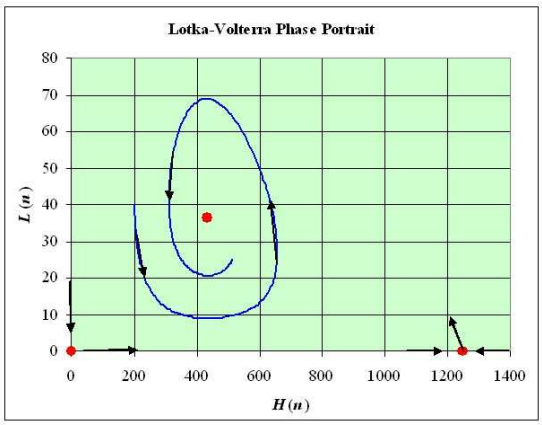
$$H_{n+1} = H_n + 0.025H_n - 0.00002H_n^2 - 0.00045H_nP_n,$$

$$P_{n+1} = P_n - 0.065P_n + 0.00015H_nP_n,$$

where n is in days. Below is the simulation for 100 days of this model.



Time simulation



Phase portrait

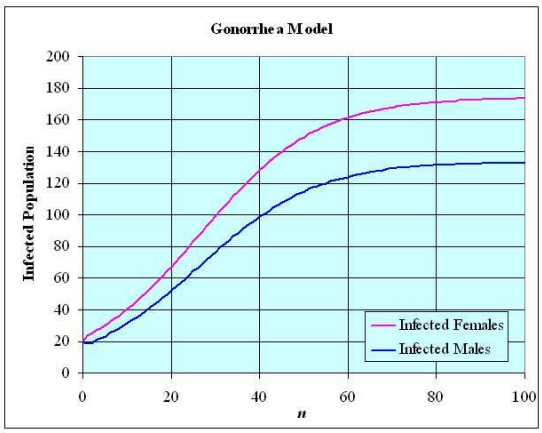
d. The equilibria for this model are $(H_e, P_e) = (0, 0)$, $(1250, 0)$ and $(433.3, 36.30)$. The equilibrium at $(H_e, P_e) = (0, 0)$ is a saddle node with eigenvalue $\lambda_1 = 1.025$ with associated eigenvector $\xi_1 = [1, 0]$ and eigenvalue $\lambda_2 = 0.935$ with associated eigenvector $\xi_2 = [0, 1]$. The equilibrium at $(H_e, P_e) = (1250, 0)$ is a saddle node with eigenvalue $\lambda_1 = 0.975$ with associated eigenvector $\xi_1 = [1, 0]$ and eigenvalue $\lambda_2 = 1.1225$ with associated eigenvector $\xi_2 = [-3.814, 1]$. The equilibrium at $(H_e, P_e) = (433.3, 36.30)$ has eigenvalues $\lambda = 0.9957 \pm 0.03229i$, which have magnitude $|\lambda| = 0.9962$. It follows that this equilibrium produces an unstable spiral with solutions moving away from this equilibrium. See graph above.

5. a. The model for gonorrhea is given by the equation:

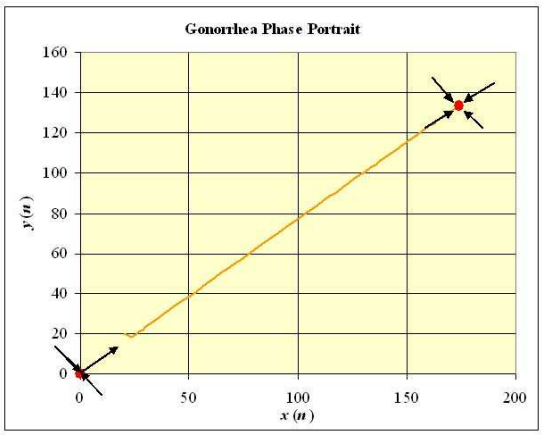
$$\begin{aligned} x_{n+1} &= x_n - a_1x_n + b_1(c_1 - x_n)y_n, \\ y_{n+1} &= y_n - a_2y_n + b_2(c_2 - y_n)x_n, \end{aligned}$$

where n is in months with x_n and y_n representing infected females and males. The equilibria for this model are $(x_e, y_e) = (0, 0)$ and $(173.97, 133.65)$. The equilibrium at $(x_e, y_e) = (0, 0)$ is a saddle node with eigenvalue $\lambda_1 = 1.0744$ with associated eigenvector $\xi_1 = [0.7894, 0.6139]$ and eigenvalue $\lambda_2 = 0.06562$ with associated eigenvector $\xi_2 = [-0.6530, 0.7590]$. The equilibrium at $(x_e, y_e) = (173.97, 133.65)$ is a stable node with eigenvalue $\lambda_1 = 0.9258$ with associated eigenvector $\xi_1 = [0.7972, 0.6038]$ and eigenvalue $\lambda_2 = 0.06290$ with associated eigenvector $\xi_2 = [-0.6244, 0.7815]$. Graphs of the times series and phase portrait are shown below. The time series shows that the female infectives rise more rapidly and achieve a higher equilibrium, which is to be expected with the disease being asymptomatic in females. Thus, women tend to harbor this disease more than men. With these parameter values, gonorrhea reaches an endemic state (nonzero) with the disease persisting in the population.

b. With $b_1 = 0.00042$ and $b_2 = 0.00037$, the equilibria are $(x_e, y_e) = (0, 0)$ and $(-70.27, -51.59)$. The equilibrium at $(x_e, y_e) = (0, 0)$ is a stable node with eigenvalue $\lambda_1 = 0.9767$ with associated eigenvector $\xi_1 = [0.8076, 0.5897]$ and eigenvalue $\lambda_2 = 0.1633$ with associated eigenvector $\xi_2 = [-0.6394, 0.7713]$. The equilibrium at $(x_e, y_e) = (-70.27, -51.59)$ is a saddle node with eigenvalue $\lambda_1 = 1.023$ with associated eigenvector $\xi_1 = [0.8047, 0.5937]$ and eigenvalue $\lambda_2 = 0.1643$ with associated eigenvector $\xi_2 = [-0.6503, 0.7629]$. It follows that the disease will

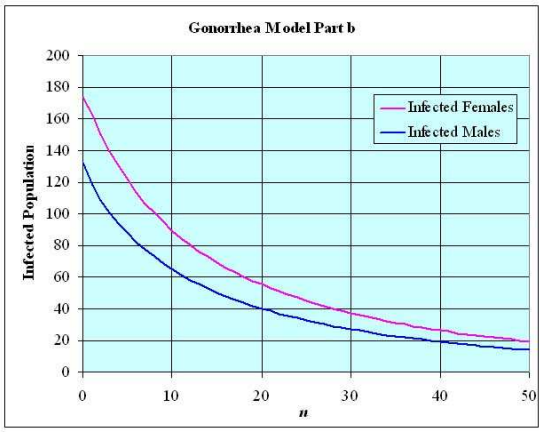


Time simulation

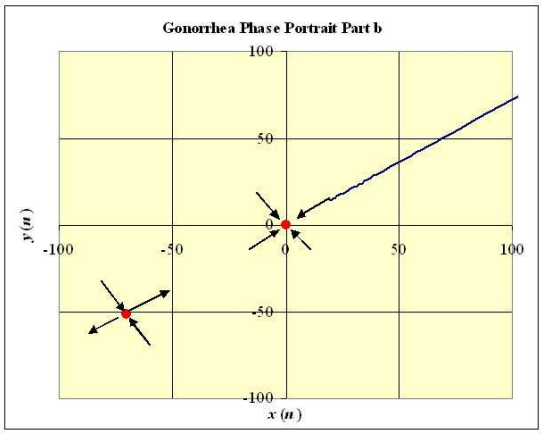


Phase portrait

go extinct as the zero equilibrium is stable. Graphs of the times series and phase portrait are shown below.



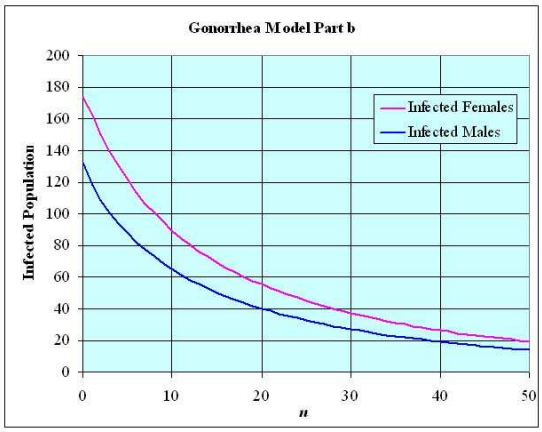
Time simulation



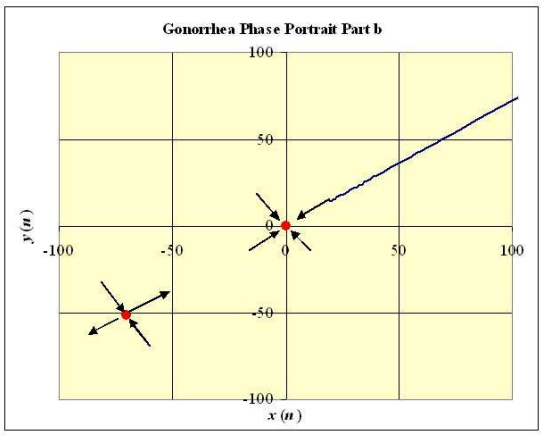
Phase portrait

c. With $a_1 = 0.43$ and $a_2 = 0.63$, the equilibria are $(x_e, y_e) = (0, 0)$ and $(-59.35, -46.33)$. The equilibrium at $(x_e, y_e) = (0, 0)$ is a stable node with eigenvalue $\lambda_1 = .9744$ with associated eigenvector $\xi_1 = [0.7894, 0.6139]$ and eigenvalue $\lambda_2 = -0.03438$ with associated eigenvector $\xi_2 = [-0.6530, 0.7590]$. The equilibrium at $(x_e, y_e) = (-59.35, -46.33)$ is a saddle node with eigenvalue $\lambda_1 = 1.026$ with associated eigenvector $\xi_1 = [0.7872, 0.6167]$ and eigenvalue $\lambda_2 = -0.033652$ with associated eigenvector $\xi_2 = [-0.6606, 0.7528]$. It follows that the disease will go extinct as the zero equilibrium is stable. Graphs of the times series and phase portrait are shown below.

d. This model shows that the disease gonorrhea could be extinguished by either sufficient treatment or decreased infectivity. This model provides an easy means to demonstrate how different treatments can effect the course of the disease gonorrhea. The parameters could readily be adjusted to match disease situations measured by data. The weaknesses of the model are the assumptions of a well-mixed population of heterosexual individuals, not accounting for complicated relations between human populations.



Time simulation



Phase portrait