

1. Solve the following linear differential equations:

a.  $\frac{dy}{dt} = 2y, \quad y(0) = 6.$

b.  $\frac{dz}{dt} = 0.1z - 2, \quad z(0) = 5.$

c.  $\frac{dx}{dt} = -\frac{x}{3}, \quad x(0) = 10.$

d.  $\frac{dh}{dx} = 5 - 0.2h, \quad h(0) = 10.$

e.  $\frac{dy}{dt} = 0.02y, \quad y(2) = 50.$

f.  $\frac{dr}{dt} = 1 - \frac{r}{4}, \quad r(1) = 6.$

2. a. A population of yeast satisfies the differential equation for Malthusian growth. If this population satisfies

$$\frac{dY}{dt} = 0.14Y,$$

with an initial population of 100 and  $t$  is in hours, then determine its population as a function of time, and find at what time the population doubles.

b. A competing population is falling, due to the presence of this new population. Suppose it satisfies the initial value problem

$$\frac{dP}{dt} = -0.07P, \quad P(0) = 1000.$$

Find the solution to this problem, and determine when its population is half of its original population.

c. Find when the two populations are equal.

3. a. The population of Canada was 24,070,000 in 1980, while in 1990 it was 26,620,000. Assuming the population is growing according to the principle of Malthusian growth (with no food or space limitations), find the population as a function of time, and determine its doubling time.

b. For the same years, the populations of Kenya were 16,681,000 and 24,229,000, respectively. Find the population of Kenya as a function of time, assuming it too is growing with Malthusian growth. What is Kenya's doubling time for its population?

c. Use these models to project the populations in the two countries in the year 2000. In what year do the populations of Canada and Kenya become equal?

4. A radioactive substance satisfies the differential equation

$$R'(t) = -kR(t)$$

for some constant  $k$ . Suppose that initially there are 10 mg of the substance. After 25 days there are 8 mg remaining. Find  $k$  and determine the half-life of the substance (time when  $R(t) = 5$ ).

5. When Strontium-90 ( $^{90}\text{Sr}$ ) is ingested, it can displace calcium in the formation of bones. After a beta decay, it becomes an isotope of krypton (an inert gas), and diffuses out of the bone, leaving the bones porous.

a. Suppose that a particular bone contains 20 mg of  $^{90}\text{Sr}$ , which has a half-life of 28 years. Write an equation describing the amount of  $^{90}\text{Sr}$  remaining at any time, and determine the amount after 10 years.

b. Find how long until only 7 mg of  $^{90}\text{Sr}$  remain.

6. Michael Crichton in the *Andromeda Strain* (1969) states that “A single cell of the bacterium *E. coli* would, under ideal circumstances, divide every twenty minutes... [I]t can be shown that in a single day, one cell of *E. coli* could produce a super-colony equal in size and weight to the entire planet Earth.” A single *E. coli* has a volume of about  $1.7 \mu\text{m}^3$ . The diameter of the Earth is 12,756 km, so assuming it is a perfect sphere, determine how long it takes for an ideally growing colony (Malthusian growth) of *E. coli* (doubling every 20 min) to equal the volume of the Earth.

7. You have just boiled a new batch of broth for your important culture of *E. coli*, so it is at  $100^\circ\text{C}$ . You have it sitting in a room that is at  $22^\circ\text{C}$ , and you find 5 minutes later that it's cooled to  $93^\circ\text{C}$ . You want to inoculate the culture when it reaches  $40^\circ\text{C}$ . You are interested in knowing if you can safely go off to exercise while the broth cools.

a. Let  $T(t)$  be the temperature of the broth. Assume that the broth satisfies Newton's Law of Cooling, and set up the differential equation for the temperature of the broth, and solve it.

b. Find how long it will be until you need to inoculate the broth with your culture. Sketch a graph of the function of  $T(t)$  for the first hour showing its starting and ending temperatures for the hour.

8. A thin plate is heated to  $100^\circ\text{C}$  for purposes of sterilization. Assume that the plate is placed into a room at  $20^\circ\text{C}$  and cools according Newton's Law of cooling, *i.e.*, the change in temperature of the plate is proportional to the difference between the temperature of the room and the temperature of the plate. After 10 min the plate is found to have a temperature of  $80^\circ\text{C}$ .

a. Write the differential equation describing the temperature of the plate,  $T(t)$ , and solve it for any time  $t \geq 0$ .

b. Find when the plate has cooled to  $30^\circ\text{C}$  so that it can be inoculated with a culture of cells.

9. a. *Paramecia* in a pond sample are growing according to the principle of Malthusian growth (with no food or space limitations). Initially, there are 1500 *Paramecia*. Four hours later, the population has 2000 individuals. Find the population of *Paramecia* as a function of time, and determine its cell doubling time.

b. A large population of 5000 is transferred to where a limited diet affects their growth dynamics. The *Paramecia* now satisfy the population dynamics of the differential equation:

$$\frac{dP}{dt} = -0.1P + 100.$$

Solve this differential equation and find what happens to the population as  $t \rightarrow \infty$ . (Hint: Recall your techniques for Newton's Law of cooling.)

10. A well mixed pond,  $V = 200,000 \text{ m}^3$ , is initially clean ( $c(0) = 0$ ). A polluted stream with a concentration of dioxin,  $Q = 5 \text{ ppb}$  enters the pond, flowing at a rate of  $f = 4000 \text{ m}^3/\text{day}$ . Another stream carries the water away at the same rate.

- a. Set up the initial value problem and solve it.
- b. Find how long before the pond has a concentration of 4 ppb.
- c. Find the limiting concentration.

11. a. An initially clean lake ( $c(0) = 0$ ) concentrates pollution from an incoming stream because of evaporative loss of water of  $200 \text{ m}^3/\text{day}$ . The well-mixed lake has a stream flowing in at a rate of  $f_1 = 2200 \text{ m}^3/\text{day}$  with a pesticide concentration of  $Q = 10 \text{ ppb}$ . The lake maintains a constant volume of  $V = 10^6 \text{ m}^3$  by having a stream leaving with a flow of  $f_2 = 2000 \text{ m}^3/\text{day}$ . Find and solve the differential equation describing the concentration of pollutant in the lake.

b. Determine how long until the lake has a concentration of 5 ppb of pesticide. Also, find the limiting concentration of pesticide. Sketch a graph of the solution.

12. Denote by  $L(t)$  the length of a fish at time  $t$  and assume that the fish grows according to the von Bertalanffy equation

$$\frac{dL}{dt} = k(34 - L(t)) \quad L(0) = 2.$$

- a. Solve this differential equation.
- b. Use your solution from Part a to determine  $k$  under the assumption that  $L(4) = 10$ . Sketch the graph of  $L(t)$  for this value of  $k$ .
- c. Find the length of the fish when  $t = 10$ . Also, find the asymptotic length of the fish, *i.e.*, find  $\lim_{t \rightarrow \infty} L(t)$ .