1. For the following systems of differential equations, find the general solution. List the eigenvalues and associated eigenvectors for the matrix in the equation. State the type of equilibrium point. Use a computer program to generate an accurate phase portrait, including the straight line solutions and at least 4 typical solutions (between the straight line solutions). Also, solve the initial value problem with

$$Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

a. $\dfrac{dY}{dt} = \begin{pmatrix} 5 & -8 \\ 4 & -7 \end{pmatrix} Y$ b. $\dfrac{dY}{dt} = \begin{pmatrix} -2 & 0 \\ 1 & -3 \end{pmatrix} Y$ c. $\dfrac{dY}{dt} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} Y$

2. Consider the following initial value problem:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = 0, \qquad y(0) = 2 \qquad y'(0) = -2.$$

a. Solve this initial value problem.

b. Create a first order system of differential equations and give the general solution. List the eigenvalues and associated eigenvectors for the matrix in the equation. State the type of equilibrium point.

c. Use a computer program to show the direction field. Include the specific solution to the initial value problem.

3. Consider the following initial value problem:

$$\begin{aligned} \frac{dx}{dt} &= -\frac{x}{t} + y, & x(1) &= -3 \\ \frac{dy}{dt} &= -2y, & y(1) &= 4 \end{aligned}$$

a. This is a decoupled system of differential equations. Find the solution to this initial value problem.

b. Use Euler's Method for 2-D systems of differential equations to approximate the solution to this initial value problem for $t \in [1, 2]$ with both $h = 0.1$ and $h = 0.05$.

c. Compare the approximate solutions and the exact solution at $t = 2$.

4. Below are two models for competition of species. Model A is given by

$$\begin{aligned}\frac{dx}{dt} &= 0.2x - 0.004x^2 - 0.008xy, \\ \frac{dx}{dt} &= 0.1y - 0.006xy - 0.002y^2,\end{aligned}$$

while Model B is given by

$$\begin{aligned}\frac{dx}{dt} &= 0.1x - 0.01x^2 - 0.005xy, \\ \frac{dx}{dt} &= 0.2y - 0.002xy - 0.004y^2.\end{aligned}$$

a. Find the equilibria for each of these models.

b. Use a computer program to generate the direction field for each of these models. Be sure that the range of the direction field includes all equilibria. Show at least 4 solutions curves that indicate what happens to the populations.

c. From the graphics of the direction field, classify each of your equilibria as saddles, sinks, or sources.

d. Discuss the differences between Model A and Model B. List the eventual outcomes in each of models. What happens when there is only a small population of x, only a small population of y, a small equal population of x and y, and a large equal population of x and y?

5. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

a. Show that if one or both of the eigenvalues of $\mathbf{A}$ is zero, then the determinant of $\mathbf{A}$ is zero.

b. Show that if the $\det\mathbf{A} = 0$, then at least one of the eigenvalues of $\mathbf{A}$ is zero.