1. a. The differential equation

$$\frac{dy}{dt} = \sin(t)y + 4\sin(t) \qquad \text{which can be written} \qquad \frac{dy}{dt} - \sin(t)y = 4\sin(t)$$

is a linear differential equation. Its integrating factor is

$$\mu(t) = exp\left(-\int \sin(t)dt\right) = e^{\cos(t)}.$$

Thus,

$$\frac{d}{dt}\left(e^{\cos(t)}y(t)\right) = 4e^{\cos(t)}\sin(t)$$

or

$$e^{\cos(t)}y(t) = 4\int e^{\cos(t)}\sin(t)dt + C = -4e^{\cos(t)} + C.$$

It follows that

$$y(t) = Ce^{-\cos(t)} - 4.$$

The initial condition $y(0) = 5$ gives $5 = Ce^{-1} - 4$ or $C = 9e$,

$$y(t) = 9e^{1-\cos(t)} - 4.$$

b. The differential equation

$$\frac{dy}{dt} = \frac{2y^2}{t+2}$$

is solved by separation of variables as follows:

$$\int \frac{dy}{y^2} = \int \frac{2dt}{t+2} \qquad \text{or} \qquad -\frac{1}{y(t)} = 2\ln|t+2| + C.$$

From the initial condition, $y(0) = 4$,

$$C = -\frac{1}{4} - 2\ln(2).$$

It follows (after some algebra) that the solution is given by

$$y(t) = \frac{4}{1 + 8\ln(2) - 8\ln(t+2)}.$$

c. The differential equation

$$\frac{dy}{dt} = -0.1y + 6 \qquad \text{or} \qquad \frac{dy}{dt} = -0.1(y - 60)$$

uses the substitution $z(t) = y(t) - 60$. Thus, we solve the differential equation

$$\frac{dz}{dt} = -0.1z, \quad z(0) = y(0) - 60 = 20 - 60 = -40.$$

Thus,

$$z(t) = -40e^{-0.1t} = y(t) - 60 \qquad \text{or} \qquad y(t) = 60 - 40e^{-0.1t}.$$

d. The differential equation

$$t\frac{dy}{dt} = 2y + t^3\cos(4t) \qquad \text{which can be written} \qquad \frac{dy}{dt} - \frac{2}{t}y = t^2\cos(4t)$$

is a linear differential equation. Its integrating factor is

$$\mu(t) = exp\left(-\int \frac{2}{t}dt\right) = e^{-2\ln|t|} = \frac{1}{t^2}.$$

Thus,

$$\frac{d}{dt}\left(\frac{y(t)}{t^2}\right) = \cos(4t)$$

or

$$\frac{y(t)}{t^2} = \int \cos(4t)dt = \frac{1}{4}\sin(4t) + C.$$

It follows that

$$y(t) = \frac{t^2}{4}\sin(4t) + Ct^2.$$

The initial condition $y(\pi) = 2$ gives $2 = C\pi^2$ or $C = \frac{2}{\pi^2}$,

$$y(t) = \frac{t^2}{4}\sin(4t) + \frac{2t^2}{\pi^2}.$$

e. The differential equation

$$\frac{dy}{dt} = -\frac{y}{3}, \quad y(2) = 15$$

has the solution

$$y(t) = 15e^{-\frac{(t-2)}{3}}.$$

f. The differential equation

$$\frac{dy}{dt} = e^{t-y}$$

is solved by separation of variables as follows:

$$\int e^y dy = \int e^t dt \qquad \text{or} \qquad e^{y(t)} = e^t + C.$$

From the initial condition, $y(0) = 3$, $C = e^3 - 1$. It follows that the solution is given by

$$y(t) = \ln\left(e^t + e^3 - 1\right).$$

2. a. The differential equation

$$\frac{dy}{dt} = t - y \qquad \text{which can be written} \qquad \frac{dy}{dt} + y = t$$

is a linear differential equation. Its integrating factor is

$$\mu(t) = exp\left(\int dt\right) = e^t.$$

Thus,

$$\frac{d}{dt}\left(e^t y(t)\right) = te^t$$

or

$$e^t y(t) = \int te^t dt + C = te^t - e^t + C.$$

It follows that

$$y(t) = t - 1 + Ce^{-t}.$$

The initial condition $y(0) = 3$ gives $3 = -1 + C$ or $C = 4$,

$$y(t) = t - 1 + 4e^{-t}.$$

b. Euler's formula for this differential equation is

$$y_{n+1} = y_n + h(t_n - y_n) = y_n + 0.5(t_n - y_n).$$

Below is a table starting with the initial condition $y_0 = 3$ and solving for $t \in [0, 2]$.

t_n	y_n
$t_0 = 0$	$y_0 = 3$
$t_1 = 0.5$	$y_1 = y_0 + 0.5(t_0 - y_0) = 3 + 0.5(0 - 3) = 1.5$
$t_2 = 1.0$	$y_2 = y_1 + 0.5(t_1 - y_1) = 1.5 + 0.5(0.5 - 1.5) = 1.0$
$t_3 = 1.5$	$y_3 = y_2 + 0.5(t_2 - y_2) = 1.0 + 0.5(1.0 - 1.0) = 1.0$
$t_4 = 2.0$	$y_4 = y_3 + 0.5(t_3 - y_3) = 1.0 + 0.5(1.5 - 1.0) = 1.25$

c. The exact solution for $t = 2$ is $y(2) = 1 + 4e^{-2} \simeq 1.54134$, so the error is $\frac{|1.54134 - 1.25|}{1.54134} \times 100 = 18.9\%$.

3. a. The differential equation

$$\frac{dy}{dt} = y + e^t \qquad \text{which can be written} \qquad \frac{dy}{dt} - y = e^t$$

is a linear differential equation. Its integrating factor is

$$\mu(t) = exp\left(-\int dt\right) = e^{-t}.$$

Thus,

$$\frac{d}{dt}\left(e^{-t} y(t)\right) = 1$$

or

$$e^{-t}y(t) = \int dt + C = t + C.$$

It follows that

$$y(t) = te^t + Ce^t.$$

The initial condition $y(0) = -1$ gives $C = -1$,

$$y(t) = (t-1)e^t.$$

b. Euler's formula for this differential equation is

$$y_{n+1} = y_n + h(y_n + e^{t_n}) = y_n + 0.5(y_n + e^{t_n}).$$

Below is a table starting with the initial condition $y_0 = -1$ and solving for $t \in [0, 2]$.

t_n	y_n
$t_0 = 0$	$y_0 = -1$
$t_1 = 0.5$	$y_1 = y_0 + 0.5(y_0 + e^{t_0}) = -1 + 0.5(-1 + e^0) = -1.0$
$t_2 = 1.0$	$y_2 = y_1 + 0.5(y_1 + e^{t_1}) = -1 + 0.5(-1 + e^{0.5}) = -0.67564$
$t_3 = 1.5$	$y_3 = y_2 + 0.5(y_2 + e^{t_2}) = -0.67564 + 0.5(-0.67564 + e^1) = 0.34568$
$t_4 = 2.0$	$y_4 = y_3 + 0.5(y_3 + e^{t_3}) = 0.34568 + 0.5(0.34568 + e^{1.5}) = 2.75937$

c. The exact solution for $t = 2$ is $y(2) = e^2 \simeq 7.38906$, so the error is $\frac{|7.38906 - 2.75937|}{7.38906} \times 100 = 62.7\%$. The Euler's solution is not close.

4. a. The differential equation $\dfrac{dy}{dt} = (y+2)(y-2)^3$ has equilibria at $y = \pm 2$ with $y = -2$ being a sink and $y = 2$ being a source.

b. The differential equation $\dfrac{dy}{dt} = 16y^2 - y^4$ has equilibria at $y = 0, \pm 4$ with $y = 4$ being a sink, $y = 0$ being a node, and $y = -4$ being a source.

c. The differential equation $\dfrac{dy}{dt} = y\sin(y)$ has equilibria at $y = n\pi$ for any integer n. The equilibrium at $y = 0$ is a node, while $y = 2k - 1$ or $-2k$ are sinks for $k = 1, 2, 3, ...$, and $y = 2k$ or $-(2k-1)$ are sources for $k = 1, 2, 3, ...$

5. a. The differential equation $\dfrac{dy}{dt} = \alpha y - 4y^3$ has a bifurcation occurring at $\alpha = 0$. (Pitchfork bifurcation) When $\alpha \leq 0$, then there is a single equilibrium at $y = 0$, and it is a sink. When $\alpha > 0$, then there are three equilibria at $y = 0, \pm\frac{\sqrt{\alpha}}{2}$. The equilibrium at $y = 0$ is a source, while the equilibria at $y = \pm\frac{\sqrt{\alpha}}{2}$ are sinks.

b. The differential equation $\dfrac{dy}{dt} = \alpha - \cosh(y)$ has a bifurcation occurring at $\alpha = 1$. (Saddle node bifurcation) When $\alpha < 1$, then there are no equilibria and all solutions tend to $-\infty$. When $\alpha = 1$ then there is a single equilibrium at $y = 0$, and it is a node. When $\alpha > 1$, then there are two equilibria at $y = \pm\cosh^{-1}(\alpha)$. The equilibrium at $y = -\cosh^{-1}(\alpha)$ is a source, while the equilibria at $y = \cosh^{-1}(\alpha)$ is a sink.

6. a. The water satisfies Newton's law of cooling with $H' = -k(H - 21)$. This differential equation is solved by setting $z(t) = H(t) - 21$. Since $H(0) = 85$, we have the shifted initial value problem

$$\frac{dz}{dt} = -kz, \qquad z(0) = 85 - 21 = 64,$$

so $z(t) = 64e^{-kt} = H(t) - 21$ or $H(t) = 64e^{-kt} + 21$. With $H(5) = 81$, we have $64e^{-5k} + 21 = 81$ or $e^{5k} = \frac{64}{60}$. Thus, $k = \frac{1}{5}\ln\left(\frac{16}{15}\right) \simeq 0.01291$. To find when $H(t) = 100$, we solve $64e^{-kt} + 21 = 100$ or $e{-}kt = \frac{79}{64}$. Thus, $t = -\frac{1}{k}\ln(\frac{79}{64}) \simeq -16.31$ min. Thus, the talks went a little over 16 min past the schedule.

b. For the tea to be at least 93°C, $64e^{-kt} + 21 = 93$ or $e{-}kt = \frac{72}{64}$. Thus, $t = -\frac{1}{k}\ln(\frac{9}{8}) \simeq -9.12$ min. So you can wait about 9 min from the escheuled end of the talks.

7. a. The differential equation with the parameters for this disease is given by

$$\frac{di}{dt} = \alpha i(1 - i) - \beta i = 0.1i(1 - i) - 0.07i = 0.03i - 0.1i^2 = 0.1i(0.3 - i).$$

It follows that the equilibria for this model are $i = 0$ and $i = 0.3$ (30% of the population infected). The equilibrium at $i = 0$ is a source, while the equilibirum at $i = 0.3$ is a sink, so all solutions tend toward $i = 0.3$, which implies that ultimately 30% of the population will be infected with this disease.

b. This problem is like the logistic growth model, so a Bernoulli's substitution is the easiest method to solve the equation. Let $z(t) = i^{-1}(t)$, so $\frac{dz}{dt} = -i^{-2}(t)\frac{di}{dt}$. If we multiply the equation above by $-i^{-2}(t)$, then

$$-i^{-2}(t)\frac{di}{dt} = -0.03i^{-1} + 0.1 \qquad \text{or} \qquad \frac{dz}{dt} = -0.03z + 0.1 = -0.03\left(z - \frac{10}{3}\right)$$

The equation is $z(t)$ is solved by another substitution, $w(t) = z(t) - \frac{10}{3}$. Since $i(0) = 0.1$, it follows that $z(0) = 10$ and $w(0) = \frac{20}{3}$. Hence,

$$w(t) = \frac{20}{3}e^{-0.03t} = z(t) - \frac{10}{3} \qquad \text{or} \qquad z(t) = \frac{20}{3}e^{-0.03t} + \frac{10}{3}.$$

Thus,

$$i(t) = \frac{3}{10 + 20e^{-0.03t}}.$$

c. Returning to the differential equation

$$\frac{di}{dt} = 0.1i(1-i) - \beta i = i\left((0.1-\beta) - 0.1i\right),$$

we see that the equilibria are $i = 0$ and $i = 1 - 10\beta$. Clearly, a bifurcation (transcritical bifurcation) occurs at $\beta = 0.1$. For $0 \leq \beta < 0.1$, the equilibrium $i = 0$ is a source and $i = 1 - 10\beta$ is a sink with the disease remaining endemic at the level of $i = 1 - 10\beta$. For $\beta \geq 0.1$, the equilibrium $i = 0$ becomes a sink and the disease vanishes. (The other equilibrium is negative and a source, so plays no role in the qualitative picture of this model.)